Functions

Class 6

2.1 Functions

- this section (re)introduces the mathematical concept of a function
- if we have two sets A and B (which could be equal)
- if every element of A is associated with exactly one element 0f B
- then we have a function from A to B
- another word for a function is a mapping
- we write

$$f : A \to B$$

- note that the right arrow symbol is the same as for logical implication, but this is in a different context

Formula Notation

- a typical way of denoting a function is with a formula

$$f(x) = x^2$$

- this is a mapping from integers to integers

$$f : Z \rightarrow Z$$

- or from reals to reals

$$f : R \rightarrow R$$

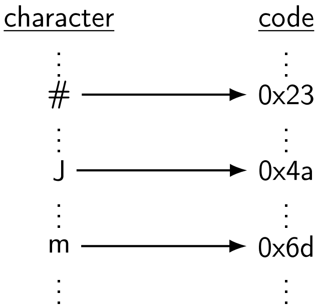
- or from naturals to naturals

$$f : N \rightarrow N$$

- the Caesar cipher is a mapping from characters to characters
- ASCII is a mapping from a particular character set to naturals, e.g., A $\rightarrow$ 65

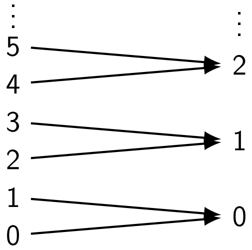
Function Digraphs

- functions are sometimes depicted as digraphs



Flavors

- if you read the definition of function very carefully
- you notice that every element of A must map to exactly one element of B
- but not every element of B must be mapped to
- and more than one element of A may map to one element of B
- what function is this?



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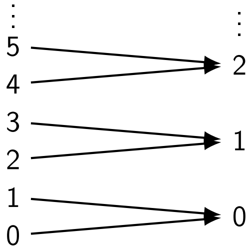
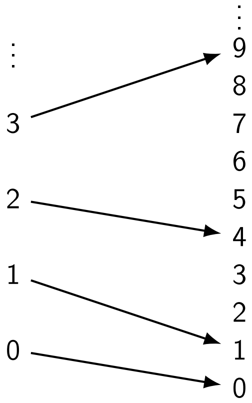


integer division by 2: $f(n) = n/2$

Another Example

- what function is this?



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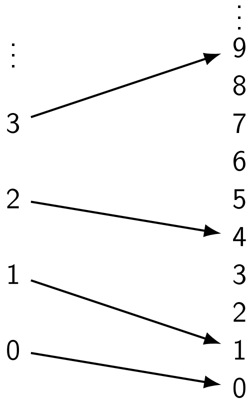

$$f(n) = n^2$$

Domain, Codomain, Range

- given a function $f : A \to B$
- the set A is the domain of the function f
- the set B is the codomain of f
- the set of B's elements that have arrows pointing to them is the range of f

- for $f : \mathrm{N} \to \mathrm{N}$ defined by $f(n) = n/2$ fill in the following:

domain:
codomain:
range:

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- for $f : \mathrm{N} \rightarrow \mathrm{N}$ defined by $f(n) = n^2$ fill in the following:

domain:	N
codomain:	N
range:	$\{0, 1, 4, 9, 16, \ldots\}$

Preimage and Image

- two terms closely related to domain and range
- the domain is all possible inputs to the function, with range being the entire set of corresponding outputs
- a preimage is an arbitrary subset of the domain
- the image is the set of outputs that correspond to whatever preimage set was supplied

Integer Functions

- some functions are extremely important in CS
- because integers and floating point numbers are so different in programming, functions that convert between them are important

- the floor function has floating point input and returns the closest integer less than or equal to the input
- floor : R $\rightarrow$ Z
- denoted floor(x) or $\lfloor x \rfloor$
  (LaTeX: `$\text{floor}(x)$` or `$\lfloor x \rfloor$`

Ceiling

- the ceiling (ceil) function is similar: returns the closest integer greater than or equal to the input

x	floor(x)	$(ceil)x$
-2	-2	-2
-1.5	-2	-1
-1.0	-1	-1
-0.5	-1	0
0	0	0
0.5	0	1
1	1	1
1.5	1	2

Quotient and Remainder

- you are familiar with integer division
- if b is not zero, then $a \div b$ gives a quotient and a remainder
- for example, $7 \div 3$ is 2 with a remainder of 1
  $6 \div 3$ is 2 with a remainder of 0

- we use the / symbol for the binary function that returns the quotient
- we use mod for the function that returns the remainder

a	b	a/b	$a \bmod b$
6	3	2	0
7	3	2	1

(LaTeX: `\bmod`)

LaTeX Delimiters

the formula for mod is:

$$a \bmod b = a - b\left\lfloor\frac{a}{b}\right\rfloor$$

notice how the floor delimiters are tall enough to reach the top and bottom of the fraction

if we do this:

$$a \bmod b = a - b\lfloor\frac{a}{b}\rfloor$$

```
\[ a \bmod b = a - b\lfloor\frac{a}{b}\rfloor \]
```

it doesn't look right

we have to do this:

```
\[ a \bmod b = a - b\left\lfloor\frac{a}{b}
  \right\rfloor \]
```

Logarithms

- logarithms are often a source of confusion
- but they are so important in CS that it's essential to get a feel for them
- in CS we deal almost exclusively with base-2 logarithms
- the simplest way to think of a logarithm is

Logarithm

How many times can you divide a number n by 2
(using integer division)
before the result is 1 or 0?

- the answer to this question is the base-2 logarithm of n

Powers of 2

here are all the powers of 2 from 0 to 10

$$2^0 = 1$$
$$2^1 = 2$$
$$2^2 = 4$$
$$2^3 = 8$$
$$2^4 = 16$$
$$2^5 = 32$$
$$2^6 = 64$$
$$2^7 = 128$$
$$2^8 = 256$$
$$2^9 = 512$$
$$2^{10} = 1024$$

Division

how many times can you divide 128 by 2 until you get 2?

Division

how many times can you divide 128 by 2 until you get 2?

7

so 7 is the base-2 logarithm of 128

Powers of 2

- here are the higher powers of 2, and their relation to powers of 10
- with these, you can interpolate

kibi	2^{10}	(1,024)	$\approx$	kilo	10^3	(one thousand)
mebi	2^{20}	(1,048,576)	$\approx$	mega	10^6	(one million)
gibi	2^{30}		$\approx$	giga	10^9	(one billion)
tebi	2^{40}		$\approx$	tera	10^{12}	(one trillion)
pebi	2^{50}		$\approx$	peta	10^{15}	(one quadrillion)
exbi	2^{60}		$\approx$	exa	10^{18}	(one quintillion)
zebi	2^{70}		$\approx$	zeta	10^{21}	(one sextillion)
yobi	2^{80}		$\approx$	yotta	10^{24}	(one septillion)

Example

about how much is 2^{37}?

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$$
\begin{aligned}
2^{37} &= 2^{30} \times 2^7 \\
&\approx 10^9 \times 2^7 \\
&\approx 1 \text{ billion} \times 128 \\
&\approx 128,000,000,000
\end{aligned}
$$

what is the approximate base-2 logarithm of 128,123,546,789?

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about 37

Logarithm Notation

$\log n$	base-10 logarithm
$\log_{10} n$	also base-10 logarithm when we wish to emphasize
$\ln n$	natural logarithm
$\lg n$	base-2 logarithm

to typeset log in LaTeX use `\log`