Function Compositions

Class 8

2.2 Composition of Functions

- this section introduces the concept of function composition

- let $g : A \rightarrow B$ be a function
- let $f : B \rightarrow C$ be a different function
- then $f \circ g$ is a composition, a new function
- $f \circ g : A \rightarrow C$

$$(f \circ g)(x) = f(g(x))$$

- composition is associative but not commutative

(LaTeX: `(f\circ g)(x) = f(g(x))`)

Example

- let $h(x) = 2x + 3$ and $g(x) = 3x + 2$
- find $(h \circ g)(x)$

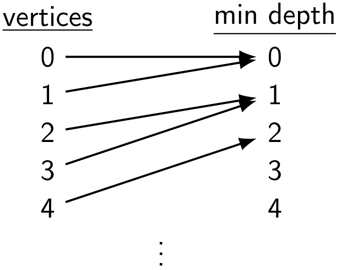
$$
\begin{aligned}
(h \circ g)(x) &= h(g(x)) \\
&= h(3x + 2) \\
&= 2(3x + 2) + 3 \\
&= 6x + 4 + 3 \\
&= 6x + 7
\end{aligned}
$$

Connection to Programming

- everything we do in this course has a direct bearing on how programs work
- discrete math is the basis of computer science, just like calculus is the basis of physics

- suppose we have a binary tree and we wish to know the minimum possible depth of the tree, given the number of vertices
- as shown in the table on page 107, there is a function that maps the number of vertices to the minimum depth

Minimum Depth



- the minimum depth function is a combination, or composition, of the floor function and the base-2 log function
- since floor and log are built into every programming language, we don't have to write our own depth function from scratch
- we can simply compose the desired function from existing pieces

Minimum Depth

- mathematically:

$$\begin{aligned}\text{min\_depth}(n) &= (\text{floor} \circ \text{lg})(n) \\ &= \text{floor}(\text{lg}(n))\end{aligned}$$

- in C++:

```
unsigned min_depth(unsigned vertices)
{
  return static_cast<unsigned>(floor(log2(vertices)));
}
```

List Functions

- four useful functions when dealing with lists are

seq	return a sequence from 0 to the argument
dist	distribute the first, singleton argument across the second, list argument, to generate a new list of tuples
pairs	return a list of pairs of corresponding elements of the two list arguments
map	similar concept to dist, but applies a function to each list element

- all four are built into many modern languages

seq: Generate a Sequence

- Python: `for i in range(4):` (gives 0, 1, 2, 3)
- BASH: `$ seq 4` (gives 1, 2, 3, 4)
- this course: $\mathrm{seq}(4) = \langle 0, 1, 2, 3, 4 \rangle$

- it's very easy to be confused by whether the list starts at 0 or 1
- and whether the end is inclusive or exclusive

dist: Distribute an Element Over a List

- in Python, this is list comprehension
- `[('a', x) for x in [1, 2, 3]]` gives `[('a', 1), ('a', 2), ('a', 3)]`
- BASH: `$ echo a{1,2,3}` gives `a1 a2 a3`
- this course: $\text{dist}(a, \langle 1, 2, 3\rangle) = \langle (a, 1), (a, 2), (a, 3)\rangle$

pairs: Create Pairs From Two Lists

- Python zip: `list(zip([1, 2, 3],['a', 'b', 'c']))` gives `[(1, 'a'), (2, 'b'), (3, 'c')]`
- BASH: `$ paste numbers letters` (assuming appropriate file contents)
- this course: $\text{pairs}(\langle 1, 2, 3\rangle, \langle a, b, c\rangle) = \langle (1, a), (2, b), (3, c)\rangle$

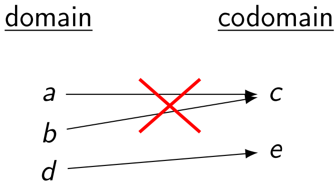
- the lists must be the same length, otherwise pairs is undefined
- in Python extra elements in one list are ignored
- in BASH extra elements are paired with the empty string

map: Apply a Function to a List

- Python: `list(map(lambda x : x**2, [0, 1, 2, 3]))` gives `[0, 1, 4, 9]`
- this course: let $L = \langle 0, 1, 2, 3 \rangle$ and $f(n) = n^2$ then $\mathrm{map}(f, L)$ gives us $\langle 0, 1, 4, 8 \rangle$

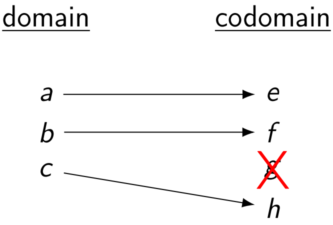
2.3 Function Characteristics

- there are two characteristics that some functions have that are extremely important
- a function is injective, or 1-to-1, if no two elements of the domain map to the same element of the range



Surjection

- a function is surjective, or "onto", if the codomain and range are the same set



Bijection

- some functions are injections but not surjections (give an example)
- some are surjections but not injections (give an example)
- some functions are neither injections nor surjections (give an example)

- but some functions are both injections and surjections
- these are called bijections, or "1-to-1 and onto"
- no two arrows point to the same element of the codomain
- every element of the codomain is pointed to by some arrow

Inverses

- bijections are so important because they are unique in being invertible
- a bijective function has an inverse function (which is also, by definition, a bijection

- if f is a function, we denote its inverse by f^{-1} (LaTeX: `$f^{-1}$`)

- for example, let f be the function $f(n) = n + 1$ which maps each odd integer to an even integer
- then $f^{-1}(n) = n - 1$ is the inverse of f which maps each even integer to an odd integer (in the same pairing as the original)

- note that $f^{-1}(f(n)) = n$ and $f(f^{-1}(n)) = n$ for any bijection f

The Pigeonhole Principle

- at the top of page 118 is a topic that is incredibly important even though it seems so obvious

The Pigeonhole Principle

If n items are put uniquely into m containers, and if $n > m$, then at least one container will have at least two items.

- your text lists some examples at the bottom of page 118, but several of these are worded so sloppily as to be incorrect

- your text incorrectly states "if a six-sided die is tossed seven times, one side will come up twice"
- the correct statement is: "if a six-sided die is tossed seven times, at least one side will come up at least twice"