Countability

Class 9

Happy
Programmer's Day!

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13 September is the 256th day of a non-leap year

2.4 Countability

- let A and B be sets
- if there is a bijection from A to B, then clearly $|A| = |B|$
- i.e., A and B have the same cardinality

- if there is an injection from A to B, then $|A| \leq |B|$

- this leads directly to an interesting fact
- is there a bijection from the natural numbers to the odd natural numbers?
- yes: $f(n) = 2n + 1$
- therefore, $|\mathrm{N}| = |\mathrm{Odd}|$
- there are exactly the same number of natural numbers as there are odd natural numbers

Countability

Countable Set

A set is countable if it is finite or if there is a bijection from it to the natural numbers.

- if there is such a bijection, the set is countably infinite
- the alternative is that the set is uncountable

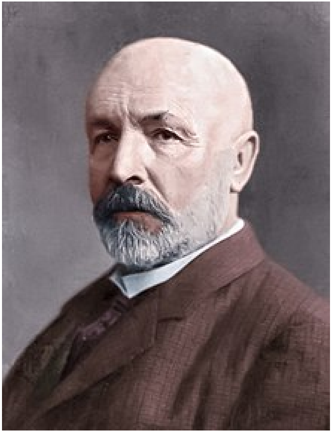
Countability

- prove that the Cartesian product $N \times N$ is countable
- to do so, we must find a bijection $N \times N \rightarrow N$
- a bijection is not obvious, but one was provided by Georg Cantor

Cantor

- Russian-born German
- 1845 – 1918
- created set theory
- defined infinity and countability
- he proved that infinity is not absolute — that there are infinities beyond infinity

- he was a devout religious Christian
- but was viciously and repeatedly attacked by theologians who believed God is infinite and nothing can be more infinite, including numbers



Cantor's Pairing

$\mathbb{N} \times \mathbb{N}$ is a set of tuples:

$$\{(0,0),(1,0),(0,1),\dots,$$
$$(2,0),(1,1),(0,2),\dots,$$
$$(3,0),(2,1),(1,2),(0,3),\dots\}$$

Cantor made up a table with three columns: tuple, sum of entries, and position, sorted by sum ascending, then by first tuple element descending

tuple	sum	position
(0, 0)	0	0
(1, 0)	1	1
(0, 1)	1	2
(2, 0)	2	3
(1, 1)	2	4
(0, 2)	2	5
(3, 0)	3	6
(2, 1)	3	7
(1, 2)	3	8
(0, 3)	3	9
(4, 0)	4	10
(3, 1)	4	11
(2, 2)	4	12
(1, 3)	4	13
(0, 4)	4	14

Cantor's Pairing

- Cantor's pairing provides a unique position value for each tuple
- the exact formula for tuple (x, y) is

$$f(x, y) = \frac{(x + y)^2 + 3y + x}{2}$$

- this is a bijection: 1-to-1 and onto
- each tuple of $N \times N$ is mapped to exactly one element of N
- and each natural number corresponds to one tuple
- therefore $N \times N$ is countable, and countably infinite

- note that Hein's position value is not exactly the same as Cantor's
- Hein orders the table by sum ascending, then by first element ascending instead of Cantor's descending

Countable Rationals

- the rational numbers are countable by the following argument (note that a rational number is always in lowest terms, so we don't have duplicates)

- the rationals can be partitioned into (i.e., are the union of) the positive rationals Q^+, the negative rationals Q^-, and zero
- since every positive rational is equivalent to a tuple in $N \times N$, we know that the positive rationals are countably infinite
- by the same argument, every negative rational is equivalent to a tuple in $N \times N$, and so the negative rationals are also countably infinite
- finally, the value zero is finite (and of size 1)
- each constituent of the partition is itself countable
- Cantor proved that the union of countable sets is countable
- therefore the rationals are countable

Uncountable Reals

- Cantor proved the real numbers are not countable
- the technique he used is called diagonalization

- consider just the real numbers between 0 and 1 and assume they are countable
- then they can be written in some order (position):

$$
\begin{aligned}
r_1 &= 0.d_{11}d_{12}d_{13}\ldots \\
r_2 &= 0.d_{21}d_{22}d_{33}\ldots \\
r_3 &= 0.d_{31}d_{32}d_{33}\ldots \\
&\vdots
\end{aligned}
$$

Uncountable Reals

- for example, let

$$\begin{aligned} r_1 &= 0.23794102\ldots \\ r_2 &= 0.44590138\ldots \\ r_3 &= 0.09118764\ldots \\ r_4 &= 0.80553900\ldots \\ &\vdots \end{aligned}$$

- note the diagonal entries highlighted in red
- now consider another rational number $R = 0.d_1 d_2 d_3 \ldots$ where the d_is are

$$d_i = \begin{cases} 4 & \text{if } d_{ii} \neq 4 \\ 5 & \text{if } d_{ii} = 4 \end{cases}$$

- the new rational number is $0.4544\ldots$
- $d_1 = 4$ because $r_{11} \neq 4$
- $d_2 = 5$ because $r_{22} = 4$, etc.

Uncountable Reals

- every real number has a unique decimal expansion
- the new real $R = 0.4544\ldots$ from the previous slide is not in the original list of r_is because R differs from r_i at each ith place
- thus there is at least one value between 0 and 1 that is not in the original list
- but this is a contradiction, so the original assumption is false

Some Things Cannot Be Computed

- every computer program is a string over the ASCII alphabet
- by a counting argument on strings, the number of computer programs is countably infinite

- by a diagonalization argument very similar to the one above, the number of different natural-number functions $f : \mathrm{N} \rightarrow \mathrm{N}$ is uncountable
- thus there are more functions than there are computer programs
- by the pigeonhole principle, some functions cannot be computed

- in a totally different way, Alan Turing also proved via the Halting Problem that some things cannot be computed