Inductively Defined Sets

Class 10

Inductive Definition

- a set can be defined by a formula
- the odd natural numbers are defined by

$$\text{Odd} = \{2n + 1 \mid n \in \mathrm{N}\}$$

- there is an alternate way to define the odd naturals:

$$\text{Odd} = \begin{cases} 1 \in \text{Odd} \\ \text{if } n \in \text{ Odd then } n+2 \in \text{Odd} \end{cases}$$

(LaTeX: `\text{Odd} = \begin{cases}`
`1 \in \text{Odd}\\`
`\text{if } n \in \text{ Odd then }n+2 \in \text{Odd}`
`\end{cases}`)

Inductive Definition

$$\text{Odd} = \begin{cases} 1 \in \text{Odd} \\ \text{if } n \in \text{ Odd then } n + 2 \in \text{Odd} \end{cases}$$

- this is an inductive set definition
- an inductive set definition consists of two parts:
  1. the base (or basis) definition or definitions
  2. the inductive rule or rules by which a new element is formed from one or more pre-existing elements
- the entire section is examples of inductively defined sets
- usually you're given the set and asked to find the bases and inductive rules
- occasionally you're given the bases and rules, and asked to find the elements

1

$S = \{3, 16, 29, 42, \ldots\}$

1

$S = \{3, 16, 29, 42, \ldots\}$

$3 \in S$; if $n \in S$ then $n + 13 \in S$

2

$$S = \{1, 3, 5, 7, \dots\}$$

2

$S = \{1, 3, 5, 7, \dots\}$

$1 \in S$; if $n \in S$ then $n + 2 \in S$

3

$$S = \{\dots, -4, -2, 0, 2, 4, \dots\}$$

3

$S = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$

$0 \in S$; if $n \in S$ then $n + 2$ and $n - 2 \in S$

4

$S = \{3, 5, 9, 17, 33, 65, \ldots\}$

4

$S = \{3, 5, 9, 17, 33, 65, \dots\}$

$3 \in S$; if $n \in S$ then $2n - 1 \in S$

5

$S = \{3, 4, 6, 7, 9, 10, 12, \dots\}$

5

$S = \{3, 4, 6, 7, 9, 10, 12, \dots\}$

3 and $4 \in S$; if $n \in S$ then $n + 3 \in S$

6

$$S = \{3, 4, 5, 8, 9, 12, 16, 17, 20, 24, 28, 32, 33, \ldots\}$$

$S = \{3, 4, 5, 8, 9, 12, 16, 17, 20, 24, 28, 32, 33, \ldots\}$

Slide $4 \cup 4 \in S$; if $n \in S$ then n

7

$S = \{n \in \mathrm{N} \mid \left\lfloor \frac{n}{2} \right\rfloor \text{ is even}\}$ (note: assume floating division)

7

$S = \{n \in \mathrm{N} \mid \left\lfloor \frac{n}{2} \right\rfloor \text{ is even}\}$ (note: assume floating division)

0 and $1 \in S$; if $n \in S$ then $n + 4 \in S$

$$S = \{n \in \mathrm{N} \mid n \bmod 5 = 2\}$$

$S = \{n \in \mathrm{N} \mid n \bmod 5 = 2\}$

$2 \in S$; if $n \in S$ then $n + 5 \in S$

9

$S = \{a^n b c^n \mid n \in \mathbb{N}\}$

$S = \{a^n b c^n \mid n \in \mathrm{N}\}$

$b \in S$; if $s \in S$ then $asc \in S$

$$S = \{a^m b^n \mid m, n \in \mathrm{N}, m > 0\}$$

$S = \{a^m b^n \mid m, n \in \mathrm{N}, m > 0\}$

$a \in S$; if $s \in S$ then $as \in S$; if $s \in S$ then $sb \in S$.

11

$S = \{a^{2n+1} \mid n \in \mathrm{N}\}$

11

$S = \{a^{2n+1} \mid n \in \mathrm{N}\}$

$a \in S$; if $s \in S$ then $saa \in S$

12

$S = \{s \in \{a, b\}^* \mid s \text{ has equal numbers of } a\text{'s and } b\text{'s}\}$

12

$S = \{s \in \{a, b\}^* \mid s \text{ has equal numbers of } a\text{'s and } b\text{'s}\}$

$\Lambda \in S$; if $s \in S$ then $sab, sba, asb, bsa, abs, bas \in S$

Even palindromes over the alphabet $\{a, b\}$

13

Even palindromes over the alphabet $\{a, b\}$

$\Lambda \in S$; if $s \in S$ then $asa, bsb \in S$

$$S = \{\langle a \rangle, \langle b, a \rangle, \langle b, b, a \rangle, \langle b, b, b, a \rangle, \ldots\}$$

14

$S = \{\langle a\rangle, \langle b, a\rangle, \langle b, b, a\rangle, \langle b, b, b, a\rangle, \ldots\}$

$\langle a\rangle \in S$; if $L \in S$ then $\text{cons}(b, L) \in S$

15

$$S = \{\langle a, b\rangle, \langle b, a\rangle, \langle a, a, b\rangle, \langle b, b, a\rangle, \langle a, a, a, b\rangle, \langle b, b, b, a\rangle, \ldots\}$$

15

$$S = \{\langle a, b\rangle, \langle b, a\rangle, \langle a, a, b\rangle, \langle b, b, a\rangle, \langle a, a, a, b\rangle, \langle b, b, b, a\rangle, \dots\}$$

Slide 14 unioned with a and b reversed

16

$S = \{L \mid L \text{ has even length over } \{a, b, c\}\}$

$S = \{L \mid L \text{ has even length over } \{a, b, c\}\}$

$\langle\rangle \in S$; if $L \in S$ then $\text{cons}(x, \text{cons}(y, L))$ where $x, y \in \{a, b, c\}$

17

$$S = \{L \in \text{lists}(\{a, b\}) \mid L \text{ has equal numbers of } a\text{'s and } b\text{'s}\}$$

17

$S = \{L \in \text{lists}(\{a, b\}) \mid L \text{ has equal numbers of } a\text{'s and } b\text{'s}\}$

$\langle\rangle \in S$
if $L \in S$ then $\text{cons}(a, \text{cons}(b, L)), \text{cons}(b, \text{cons}(a, L)),$
$\text{consR}(a, \text{consR}(b, L)), \text{consR}(b, \text{cons}(a, L)) \in S$

18

The set of all binary trees over $A = \{0, 1\}$ such that if a node is 1, its children are 0's, and if a node is 0 its children are 1's. (Note: unclear whether the empty tree is allowed; we'll assume not.)

The set of all binary trees over $A = \{0, 1\}$ such that if a node is 1, its children are 0's, and if a node is 0 its children are 1's. (Note: unclear whether the empty tree is allowed; we'll assume not.)

$\text{tree}(<>, 0, <>), \text{tree}(<>, 1, <>) \in S$
if $T_1, T_2 \in S$ and $\text{root}(T_1) = \text{root}(T_2) = 0$ then $\text{tree}(T_1, 1, T_2) \in S$
if $T_1, T_2 \in S$ and $\text{root}(T_1) = \text{root}(T_2) = 1$ then $\text{tree}(T_1, 0, T_2) \in S$