Binary Relations

Class 26

Introduction

- a binary relation is like a function in which the domain and the codomain are the same set
- if A is a set, then $A \times A$ is the set of all possible tuples of the form (x, y) where x and y are both elements of A
- a relation R over A is a subset of these tuples

Example

- let $A = \{a, b, c\}$
- then

$$A \times A = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$$

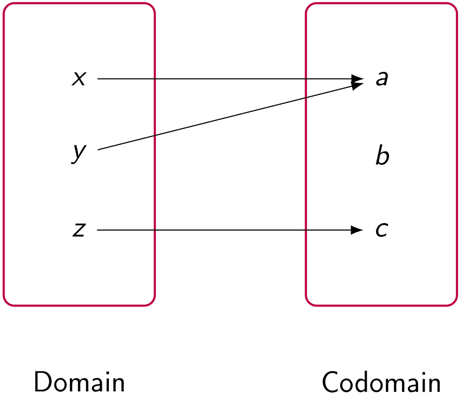
- let R be the relation "is less than" ($<$) over A
- then $R = \{(a, b), (a, c), (b, c)\}$ which is a subset of $A \times A$

Notation

- for any relation R, we can write any of these
  - $R(a, b)$
  - $a \, R \, b$
  - $(a, b) \in R$
- in the case of "is less than", these become
  - $< (a, b)$
  - $a < b$
  - $(a, b) \in <$

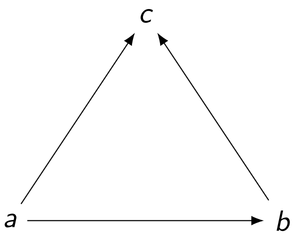
Digraph

- when we draw functions as graphs, they look like this



Relation Digraph

- but since a relation as identical domain and codomain, we draw relations like this:



Alternate Forms

- thus there are three main ways we can specify a relation
    1. a set of tuples, e.g., $\{(a, b), (a, c), (b, c)\}$
    2. a rule, e.g., $\{(x, y) \mid x, y \in A \text{ and } x < y\}$
    3. a digraph, as on the previous slide

Properties

- there are five properties that every relation either has or does not have
- you have to memorize and know these properties
- a relation is
  1. reflexive if <u>every</u> element in A is related to itself

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  5. antisymmetric if $a\ R\ b$ and $b\ R\ a$ then a and b are identical

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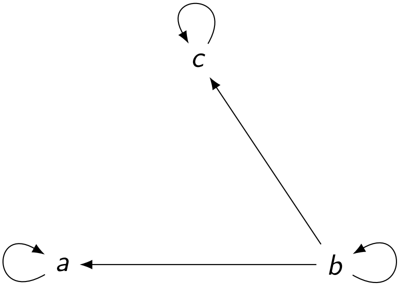
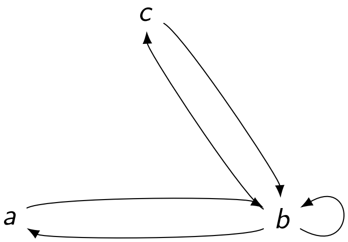
  5. antisymmetric if $a\ R\ b$ and $b\ R\ a$ then a and b are identical
- properties 1, 2, and 3 are usually thought of as positive properties, in that they specify things that must exist if the relation has that property
- properties 4 and 5 are negative properties that specify things that a relation cannot have if it has that property

Visualize Reflexive



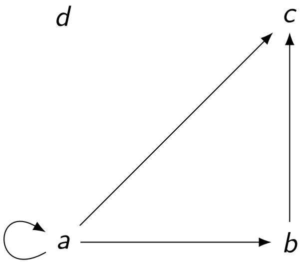
- $(a, a), (b, b), \ldots$ for every element
- every element has a self-loop, regardless of other arrows

Visualize Symmetric



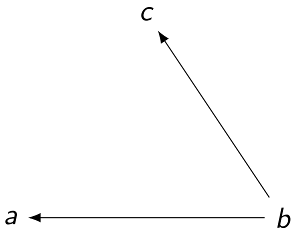
- every arrow except a self-loop is paired with one going the other way

Visualize Transitive



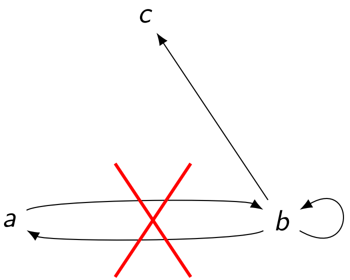
- easy to understand but hard to spot
- every time there's $x \to y$ and $y \to z$ there's also a $x \to z$

Visualize Irreflexive



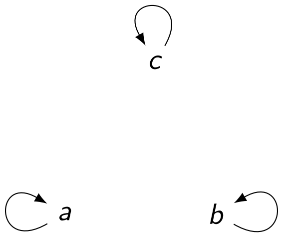
- there are no self-loops, regardless of other arrows

Visualize Antiymmetric



- nowhere is there a complementary pair, regardless of other arrows

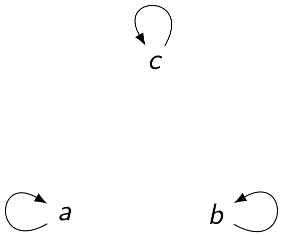
Equality



- reflexive?
- symmetric?
- transitive?

- irreflexive?
- antisymmetric?

Equality



- reflexive? yes $x = x$ for every element
- symmetric?
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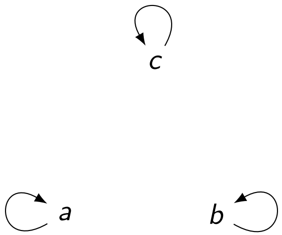
Equality



- reflexive? yes $x = x$ for every element
- symmetric? yes if $x = x$ then $x = x$
- transitive?

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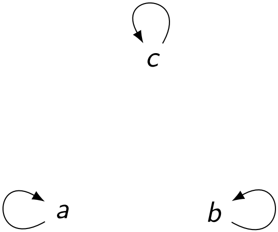
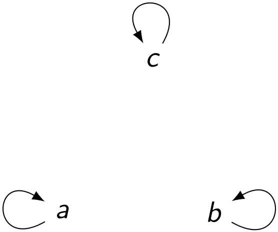
Equality



- reflexive? yes $x = x$ for every element
- symmetric? yes if $x = x$ then $x = x$
- transitive? yes every time $x = y$ and $y = z$ then $x = z$
- irreflexive?
- antisymmetric?

Equality



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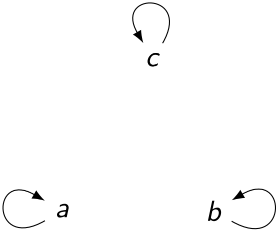
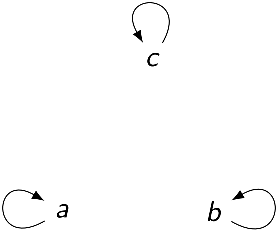
- symmetric? yes if $x = x$ then $x = x$
- transitive? yes every time $x = y$ and $y = z$ then $x = z$
- irreflexive? no a single self-loop kills it
- antisymmetric?

Equality



- reflexive? yes $x = x$ for every element
- symmetric? yes if $x = x$ then $x = x$
- transitive? yes every time $x = y$ and $y = z$ then $x = z$
- irreflexive? no a single self-loop kills it
- antisymmetric? yes there are no complementary pairs

Equality



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- transitive? yes every time $x = y$ and $y = z$ then $x = z$
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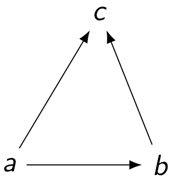
- antisymmetric? yes there are no complementary pairs

- note that a relation can be both symmetric and antisymmetric

Less Than



- reflexive?
- symmetric?
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- irreflexive?
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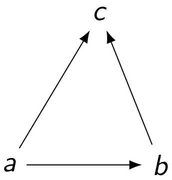
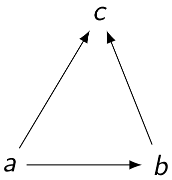
Less Than



- reflexive? no $x \not< x$ for every element
- symmetric?
- transitive?

- irreflexive?
- antisymmetric?

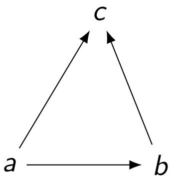
Less Than



- reflexive? no $x \not< x$ for every element
- symmetric? no if $x < y$ then $y \not< x$
- transitive?

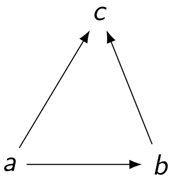
- irreflexive?
- antisymmetric?

Less Than



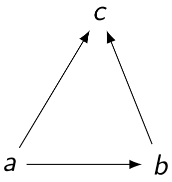
- reflexive? no $x \not< x$ for every element
- symmetric? no if $x < y$ then $y \not< x$
- transitive? yes every time $x < y$ and $y < z$ then $x < z$
- irreflexive?
- antisymmetric?

Less Than



- reflexive? no $x \nless x$ for every element
- symmetric? no if $x < y$ then $y \nless x$
- transitive? yes every time $x < y$ and $y < z$ then $x < z$
- irreflexive? yes no self-loops
- antisymmetric?

Less Than



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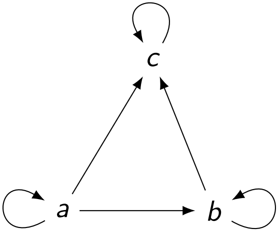
- antisymmetric? yes there are no complementary pairs

Less Than Or Equal To



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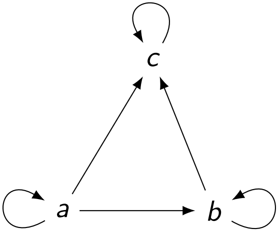

- reflexive? yes $x \leq x$ for every element
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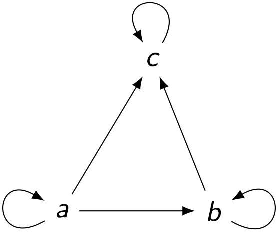

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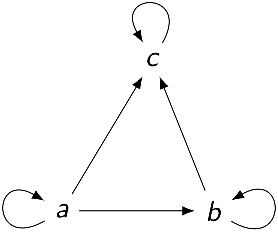

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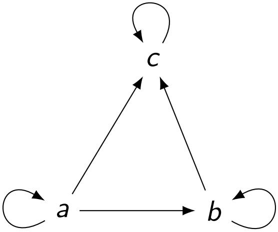

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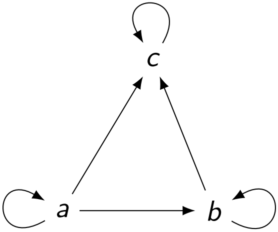

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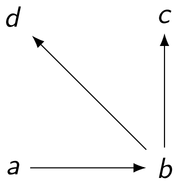
Is Parent Of



- reflexive?
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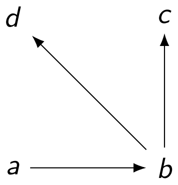

- reflexive? no no one is their own parent
- symmetric?
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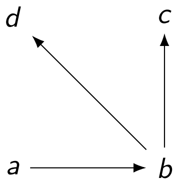
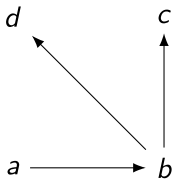


- reflexive? no no one is their own parent
- symmetric? no if Ann is Bob's parent, then Bob cannot be Ann's parent
- transitive?

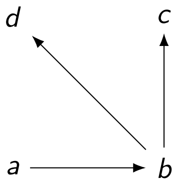
- irreflexive?
- antisymmetric?

Is Parent Of



- reflexive? no no one is their own parent
- symmetric? no if Ann is Bob's parent, then Bob cannot be Ann's parent
- transitive? no if Ann is Bob's parent and Bob is Carol's parent, then Ann is not Carol's parent
- irreflexive?
- antisymmetric?

Is Parent Of



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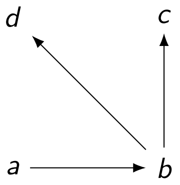

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- symmetric? no if Ann is Bob's parent, then Bob cannot be Ann's parent
- transitive? no if Ann is Bob's parent and Bob is Carol's parent, then Ann is not Carol's parent
- irreflexive? yes no one is their own parent
- antisymmetric? yes same as symmetric

Composition

- relations can be composed just like other functions
- but for relations we think of them a little differently

- let R and S be binary relations
- R is a set of tuples, and S is is a set of tuples
- then $R \circ S$ is the set of tuples (a, c) for all $(a, b) \in R$ and $(b, c) \in S$

Composition

- let $A = \{a, b, c, d\}$
- let R be the relation "less than" and S be the relation "equal to"
- what is in R and S individually?
- $R = \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}$
- $S = \{(a, a), (b, b), (c, c), (d, d)\}$

- what is in $R \circ S$?

Composition

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- what is in $R \circ S$?
- $R \circ S = \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}$

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- $S = \{(a, a), (b, b), (c, c), (d, d)\}$

- what is in $R \circ S$?
- $R \circ S = \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}$

- what is in $S \circ R$?

Composition

- let $A = \{a, b, c, d\}$
- let R be the relation "less than" and S be the relation "equal to"
- what is in R and S individually?
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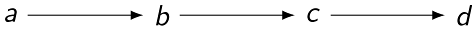
- what is in $R \circ S$?
- $R \circ S = \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}$

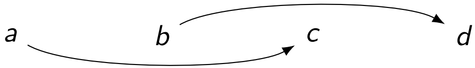
- what is in $S \circ R$?
- $S \circ R = \{(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)\}$

Self Composition

- a relation can always be composed with itself
- let $R = \{(a, b), (b, c), (c, d)\}$



- then $R \circ R = R^2 = \{(a, c), (b, d)\}$



- $R \circ R \circ R = R^3 = \{(a, d)\}$

R^0

- to make the definitions all work, we define

$$R^0 = \{(x, x) \mid x \in A\}$$

- it doesn't really make a lot of sense
- but it makes things work

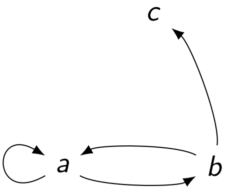
Closure

- the three “positive” properties are interesting:
- if relation R does not have the property, we can add tuples to R until it does
- this is the concept of the closure of a relation

Closure

- let R over $A = \{a, b, c\}$ be

$$\{(a, a), (a, b), (b, a), (b, c)\}$$



- reflexive?
- symmetric?
- transitive?

Closure

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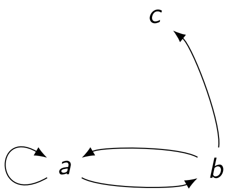
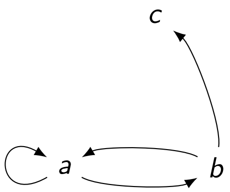


- reflexive? no
- symmetric?
- transitive?

Closure

- let R over $A = \{a, b, c\}$ be

$$\{(a, a), (a, b), (b, a), (b, c)\}$$

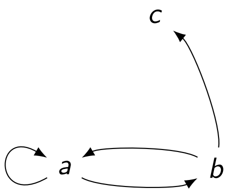


- reflexive? no
- symmetric? no
- transitive?

Closure

- let R over $A = \{a, b, c\}$ be

$$\{(a, a), (a, b), (b, a), (b, c)\}$$



- reflexive? no
- symmetric? no
- transitive? no

Closure

- let R over $A = \{a, b, c\}$ be

$$\{(a, a), (a, b), (b, a), (b, c)\}$$

- why is R not reflexive?
- what needs to be added to R to make it be reflexive?

Closure

- let R over $A = \{a, b, c\}$ be

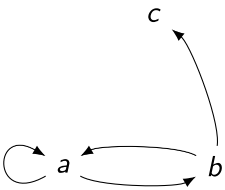
$$\{(a, a), (a, b), (b, a), (b, c)\}$$

- why is R not reflexive?
- what needs to be added to R to make it be reflexive?
- (b, b) and (c, c)
- so add them:

$$R \cup \{(b, b), (c, c)\} = r(R)$$

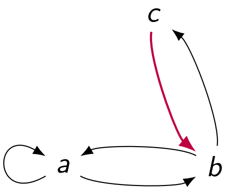
- $r(R)$ is the reflexive closure of R

Symmetric Closure



- why is R not symmetric?
- what needs to be added to R to make it be symmetric?

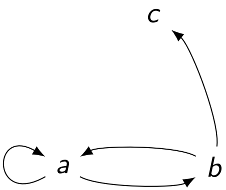
Symmetric Closure



- why is R not symmetric?
- what needs to be added to R to make it be symmetric?
- (c, b), so add it

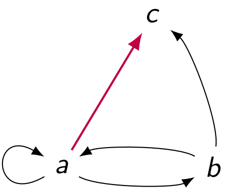
$$s(R) = R \cup \{(c, b)\}$$

Transitive Closure



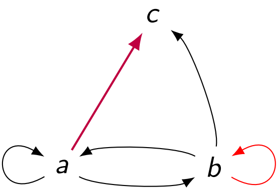
- why is R not transitive?

Transitive Closure



- why is R not transitive?
- (a, c)

Transitive Closure

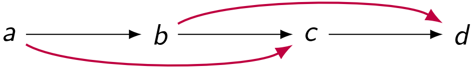


- why is R not transitive?
- (a, c)
- and (b, b)!

Transitive Closure

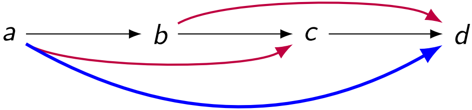
$$a \longrightarrow b \longrightarrow c \longrightarrow d$$

Transitive Closure



- need to add (a, c) and (b, d)

Transitive Closure



- need to add (a, c) and (b, d)
- oops, now we need to add (a, d)
- now it's transitive

- every time we add an edge, that might require more edges
- for $|A| = n$ there could be n rounds of adding edges
- if A is infinite, the process of creating transitive closure could be infinite

Multiple Closures

- we can create double and triple closures
- $rs(R)$ is the reflexive closure of the symmetric closure of R

- we will not do page 214 Path Problems to the end of section 4.1.