Equivalence Relations

Class 27

Introduction

- humans categorize things
- we would drown in details if we did not (in fact, an inability to categorize is one recognized form of insanity)
- a relation is called an equivalence if it is reflexive, symmetric, and transitive: RST

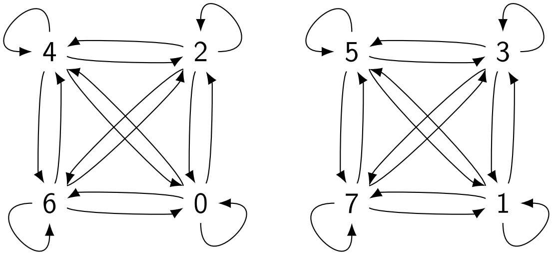
- we use the symbol R for an arbitrary relation
- we use the symbol $\sim$ for an arbitrary equivalence relation (LaTeX: `\sim`)

- the most obvious equivalence relation is simple numeric equality

$$R = \{(x, x) \mid x = x\}$$

Evenness Equivalence

- there are many other equivalence relations than equality, however
- for the set of integers, let $x \sim y$ mean x and y have the same evenness
- this means all the even integers are mutually equivalent to each other, and all the odd integers are mutually equivalent to each other
- confirm this is RST



Equivalence Examples

- for integers, $x \sim y$ means $x^2 = y^2$
- for people, $x \sim y$ means x and y have the same biological mother
- for rationals, $x \sim y$ means $x - y$ is an integer
- for binary trees, $x \sim y$ means x and y have the same number of nodes

Intersection of Equivalences

- an observation:
- let S and T be two different equivalence relations on the same set
- then $R = S \cup T$ is guaranteed to also be an equivalence relation

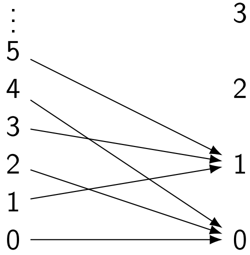
- let $S = \{(x, y) \mid x \text{ and } y \text{ have the same first name}\}$
- let $T = \{(x, y) \mid x \text{ and } y \text{ have the same last name}\}$

- S and T are both equivalence relations

- then
  $S \cup T = \{(x, y) \mid x \text{ and } y \text{ have the same first and last names}\}$
  is also an equivalence relation

Kernel Relations

- every function generates an equivalence relation on the function's domain
- for example let $f(n) = n \bmod 2$ on the domain of N



- so we have

$$R = \{(x, y) \mid x, y \in \mathrm{N} \text{ and } x \bmod 2 = y \bmod 2\}$$

- in words, $x\,R\,y$ if $f(x)$ and $f(y)$ point to the same place in f's codomain

Kernel Relations

- for any R where $x \, R \, y$ if $f(x)$ and $f(y)$ point to the same place in f's codomain, R is guaranteed to be an equivalence relation
- mathematicians use the phrase "kernel" to mean something that generates something
- in this context f is a function that generates an equivalence relation
- the relation that f generates is guaranteed to be an equivalence relation
- the relation is called the kernel relation of f

- we can sometimes use this kernel concept to prove that a relation is in fact RST, an equivalence relation

Using a Kernel Relation

- suppose we are given R on integers defined as $x \; R \; y$ iff $x - y$ is even
- prove that R is an equivalence relation, i.e., that R is RST

$x \; R \; y$	iff $x - y$ is even	given
	iff x, y both are even or odd	properties of integers
	iff $x \bmod 2 = y \bmod 2$	definition of even and odd

- therefore R is an equivalence relation because it's the kernel relation of the function $f(n) = n \bmod 2$

Equivalence Classes

- a natural part of equivalence relations is the concept of equivalence classes
- all the things in a set that are mutually equivalent form a class
- all the equivalence classes form a partition of a set: a set of nonempty subsets that are mutually disjoint and whose union is the whole set

- let $x \; R \; y$ be the relation "x is a phone number with the same area code as y"
- what are the equivalence classes of R?

- phone numbers 201-200-0000 through 201-299-9999 are in one equivalence class
- phone numbers 202-200-0000 through 202-299-9999 are in another equivalence class

Equivalence Class Name

- consider the equivalence relation over naturals defined by the function "mod2"
- what are the equivalence classes?

- $0, 2, 4, \ldots$ are all members of one equivalence class
- $1, 3, 5, \ldots$ are all members of the other equivalence class

- it would be useful to have a name for the different equivalence classes
- what do we call the class $0, 2, 4, \ldots$? obviously, in this case, we could call it the "evens"
- but in general (like the area codes) how do we refer to an equivalence class?

Equivalence Class Terminology

- $[a]$ is the name for the class of everything equivalent to a
- so a name for the evens is $[0]$
- or $[2]$ or $[100]$
- a name for the class of phone numbers starting with 202 is [202-200-0000]

- a is called the canonical name of the class
- a can be any element, but should be obvious

- for the odds, $[1]$ is the only real choice
- for the evens, either $[0]$ or $[2]$ would make sense

Kernel Recap

- let $f : Z \to N$ be $f(x) = |x|$
- what are the equivalence classes of the kernel relation of f?

Kernel Recap

- let $f : Z \to N$ be $f(x) = |x|$
- what are the equivalence classes of the kernel relation of f?

$$\begin{aligned}
[0] &= \{0\} \\
[1] &= \{-1, 1\} \\
[2] &= \{-2, 2\} \\
[3] &= \{-3, 3\} \\
&\vdots
\end{aligned}$$

Software Testing

```
function p(int x)
{
  if (x > 0)
  {
    foo(x);
  }
  else if (x % 2 == 0)
  {
    bar(x);
  }
  else
  {
    bam(x);
  }
}
```

- equivalence classes are extensively used in software engineering to create test cases for code
- p's input set is essentially infinite, so we can't test all possible inputs
- but the if-else structure partitions p's input into three equivalence classes: the positives, the negative odds, and the negative evens plus 0
- if we pick one element of each equivalence class, we will execute all pathways of p