Combinations and Permutations

Class 39

Introduction

- the big picture is counting operations performed as an algorithm executes
- counting things requires thinking about permutations and combinations

- permutation: an arrangement in which order matters e.g., lists, strings
- combination: an arrangement in which order does not matter e.g., sets

Permutations

- permutations are arrangements of things: orderings or re-orderings
- let $S = \{a, b, c\}$
- how many ways to order the three letters?
- abc, acb, bac, bca, cab, cba there are six ways

- there are 3 ways to choose the first letter
- for each of those 3 way, there are 2 ways to choose the second letter
- for each of the 3×2 ways of choosing the first two, there is 1 way to choose the last letter
- therefore there are $3 \times 2 \times 1 = 3!$ permutations of three distinct things

Permutations

- let there be a set of n distinct things and we
  1. select r of them (without replacement) and then
  2. generate all permutations of the r selected things
- how many different ways to do steps 1 and 2?

- the total number of different permutations of r things chosen from a set of n things is

$$P(n, r) = \frac{n!}{(n-r)!}$$

Combinations

- we have seen that there are $3! = 6$ ways to permute, or arrange, 3 things a, b, c
- but if we combine 3 things into one set where order doesn't matter, we just get the single combination set $\{a, b, c\}$
- so, there are fewer combinations than permutations

Combinations

- let there be a set of n distinct things and we
    1. select r of them (without replacement) and combine them into a set

- the total number of different combinations of r things chosen from a set of n things is

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

- comparing this to $P(n, r)$, you see that this denominator is bigger, so the number of combinations is smaller
- $C(n, r)$ is also written as $\binom{n}{r}$ and pronounced as "n choose r"

(LaTeX: `$\binom{n}{r}$`)

Bags

- we have not talked very much about bags so far in this course
- they have been present in the text, but we've mostly skipped the things that include them

Bag

A bag, also known as a multiset, is an unordered collection of elements in which repeated elements are allowed.

- every set is also a bag
- a bag without repeated elements is a set

Without Replacement

- the above discussion of combinations and permutations was strictly in the context of sets
- they have no repeated elements, because you can only take the same element out of a set once, not twice
- the term for this is without replacement

Permutations of a Bag

- but suppose we have a bag:

$$B = [a, a, b, b, c, c]$$

  nine 2-element permutations of this bag exist:

$$aa, ab, ac, ba, bb, bc, ca, cb, cc$$

- the number of distinct permutations of size r taken from a bag with n distinct elements is n^r
- in this example, $n = 3$ and $r = 2$, so $n^r = 3^2 = 9$

- note that even though there are two b's in the bag, there's only one permutation ab because we can't tell the two b's apart
- if we could tell them apart, e.g., b_1 and b_2, then we would have two distinct permutations ab_1 and ab_2
- when we say an element is repeated, it means the repeats cannot be distinguished

Permutations of a Bag

- generalizing, let B be a bag with n total elements and i distinct elements
- let m_1 be the number of occurrences of one of the i distinct elements
- let m_2 be the number of occurrences of another of the i distinct elements, and so on, up to m_i
- then the number of (distinct) permutations of the n elements of B is

$$\frac{n!}{m_1! \times m_2! \times \cdots \times m_i!}$$

Permutations of a Bag

- how many permutations of B's elements are there?

$$B = [a, a, b, b, b]$$

n	5
i	2 (a and b)
m_1	2 (a's)
m_2	3 (b's)

- thus there are 10 permutations:

$$\frac{5!}{2!3!} = 10$$

Combinations With Replacement

- in the earlier discussion, combinations are sets of things
- but with repeated elements, combinations become bags of things

- the number of k-element bags chosen from an n-element set is

$$\binom{n+k-1}{k}$$

Combinations with Replacement

- let there be a jar filled with a collection of many pennies, nickels, and dimes
- the number of each type of coin is essentially unlimited
- how many different ways are there of selecting 5 coins at random from the jar?

- n represents the number of different types of coin, which is 3 (penny, nickel, dime)
- k is the number of coins being selected, which is 5

$$\binom{3+5-1}{5} = \binom{7}{3}$$
$$= 35$$

Combinations with Replacement

- another way of thinking about this is to let there be exactly 3 coins in the jar instead of an unlimited number
- then, for 5 iterations, a coin is chosen from the jar, recorded, and then returned to the jar
- that is why this is selection with replacement