Section 6.2

Propositional Calculus

- *Propositional calculus* is the language of *propositions* (statements that are true or false).
- We represent propositions by formulas called *well-formed formulas* (wffs) (pronounced like woofs) that are constructed from an alphabet consisting of:
  - Truth symbols: T (or true) and F (or false)
  - Propositional variables: uppercase letters
  - Connectives (operators):
    - $\neg$ (not, negation)
    - $\wedge$ (and, conjunction)
    - $\vee$ (or, disjunction)
    - $\rightarrow$ (conditional, implication)
  - Parentheses symbols: (and).

More About Wffs

- A *wff* is either a truth symbol, a propositional variable, or if V and W are wffs, then so are:
  - $\neg V$
  - $V \wedge W$
  - $V \vee W$
  - $V \rightarrow W$
  - (W)
- Example: The expression $A \neg B$ is not a wff. But the following are wffs:
  - $A \wedge B \rightarrow C$
  - $(A \wedge B) \rightarrow C$
  - $A \wedge (B \rightarrow C)$

Truth Tables

- The connectives are defined by the following truth table.

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \to Q$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	T	T
F	F	T	F	F	T

Semantics

- The meaning of T (or True) is true and the meaning of F (or False) is false. The meaning of any other wff is its truth table, where, in the absence of parentheses, we define the hierarchy of evaluation to be $\neg$, $\wedge$, $\vee$, $\rightarrow$, and we assume $\wedge$, $\vee$, $\rightarrow$ are left associative.
- Examples:

$$\begin{array}{rcl} \neg A \wedge B & \text{means} & (\neg A) \wedge B \\ A \vee B \wedge C & \text{means} & A \vee (B \wedge C) \\ A \wedge B \rightarrow C & \text{means} & (A \wedge B) \rightarrow C \\ A \rightarrow B \rightarrow C & \text{means} & (A \rightarrow B) \rightarrow C \end{array}$$

Three Classes

- A *Tautology* is a wff for which all truth table values are T.
- A *Contradiction* is a wff for which all truth table values are F.
- A *Contingency* is a wff that is neither a tautology nor a contradiction.
- Examples:
  - $P \vee \neg P$ is a tautology.
  - $P \wedge \neg P$ is a contradiction.
  - $P \rightarrow Q$ is a contingency.

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- Example: $A \vee \neg A \equiv B \vee \neg B$

Equivalence and Tautologies

- We can express equivalence in terms of tautologies as follows:
- $V \equiv W$ iff $(V \rightarrow W)$ and $(W \rightarrow V)$ are tautologies.
- Proof: $V \equiv W$ iff V and W have the same truth values iff $(V \rightarrow W)$ and $(W \rightarrow V)$ are tautologies. QED.

Basic Equivalences that Involve True and False

The following equivalences are easily checked with truth tables:

- $A \wedge \text{True} \equiv A$
- $A \wedge \text{False} \equiv \text{False}$
- $A \wedge \neg A \equiv \text{False}$
- $A \vee \text{True} \equiv \text{True}$
- $A \vee \text{False} \equiv A$
- $A \rightarrow \text{True} \equiv \text{True}$
- $A \rightarrow \text{False} \equiv \neg A$
- $A \vee \neg A \equiv \text{True}$
- $\text{True} \rightarrow A \equiv A$
- $\text{False} \rightarrow A \equiv \text{True}$
- $A \rightarrow A \equiv \text{True}$

Other Basic Equivalences

The connectives $\wedge$ and $\vee$ are commutative, associative, and distribute over each other. These properties and the following equivalences can be checked with truth tables

- $A \wedge A \equiv A$
- $A \vee A \equiv A$
- $\neg\neg A \equiv A$
- $\neg(A \wedge B) \equiv \neg A \vee \neg B$
- $\neg(A \vee B) \equiv \neg A \wedge \neg B$
- $A \rightarrow B \equiv \neg A \vee B$
- $A \wedge (A \vee B) \equiv A$
- $A \vee (A \wedge B) \equiv A$
- $\neg(A \rightarrow B) \equiv A \wedge \neg B$
- $A \wedge (\neg A \vee B) \equiv A \wedge B$
- $A \vee (\neg A \wedge B) \equiv A \vee B$

Using Equivalences to Prove Other Equivalences

We can often prove an equivalence without truth tables because of the following two facts.

1. If $U \equiv V$ and $V \equiv W$, then $U \equiv W$.
2. If $U \equiv V$, then any wff W that contains U is equivalent to the wff obtained from W by replacing an occurrence of U by V.

continued

- Example: Use equivalences to show that $A \vee B \to A \equiv B \to A$
- Proof:

$$\begin{aligned} A \vee B \to A &\equiv \neg(A \vee B) \vee A \\ &\equiv (\neg A \wedge \neg B) \vee A \\ &\equiv (\neg A \vee A) \wedge (\neg B \vee A) \\ &\equiv \text{True} \wedge (\neg B \vee A) \\ &\equiv \neg B \vee A \\ &\equiv B \to A \end{aligned}$$

Practice Problems

Use known equivalences.

- Prove that $A \vee B \to C \equiv (A \to C) \wedge (B \to C)$.
- Prove that $(A \to B) \vee (\neg A \to B)$ is a tautology (i.e., show it is equivalent to true).
- Prove that $A \to B \equiv (A \wedge \neg B) \to \text{False}$.

Is it a tautology, a contradiction, or a contingency?

If P is a variable in a wff W, let $W(P/\text{ True})$ denote the wff obtained from W by replacing all occurrences of P by True. $W(P/\text{ False })$ is defined similarly. The following properties hold:

- W is a tautology iff $W(P/\text{ True })$ and $W(P/\text{ False })$ are tautologies.
- W is a contradiction iff $W(P/\text{ True })$ and $W(P/\text{ False })$ are contradictions.

Quine's method uses these properties together with basic equivalences to determine whether a wff is a tautology, a contradiction, or a contingency.

Example of Quine's Method

Let $W = (A \wedge B \rightarrow C) \wedge (A \rightarrow B) \rightarrow (A \rightarrow C)$.

$$\begin{aligned} W(A/\text{False}) &\equiv (\text{False} \wedge B \rightarrow C) \wedge (\text{False} \rightarrow B) \rightarrow (\text{False} \rightarrow C) \\ &\equiv (\text{False} \rightarrow C) \wedge \text{True} \rightarrow \text{True} \\ &\equiv \text{True}. \end{aligned}$$

$$\begin{aligned} W(A/\text{True}) &\equiv (\text{True} \wedge B \rightarrow C) \wedge (\text{True} \rightarrow B) \rightarrow (\text{True} \rightarrow C) \\ &\equiv (B \rightarrow C) \wedge B \rightarrow C \end{aligned}$$

Let $X = (B \rightarrow C) \wedge B \rightarrow C$. Then we have:

$$\begin{aligned} X(B/\text{True}) &\equiv (\text{True} \rightarrow C) \wedge \text{True} \rightarrow C \\ &\equiv C \wedge \text{True} \rightarrow C \\ &\equiv C \rightarrow C \\ &\equiv \text{True} \\ X(B/\text{False}) &\equiv (\text{False} \rightarrow C) \wedge \text{False} \rightarrow C \\ &\equiv \text{False} \rightarrow C \\ &\equiv \text{True} \end{aligned}$$

Practice Quine's Method

- Show that $(A \vee B \to C) \vee A \to (C \to B)$ is NOT a tautology.
- Show that $(A \to B) \to C$ is NOT equivalent to $A \to (B \to C)$.

Normal Forms

- A *literal* is either a propositional variable or its negation, e.g., A and $\neg A$ are literals.
- A *disjunctive normal form (DNF)* is a wff of the form $C_1 \vee \ldots \vee C_n$ where each C_i is a conjunction of literals, called a *fundamental conjunction*.
- A *conjunctive normal form (CNF)* is a wff of the form $D_1 \wedge \ldots \wedge D_n$ where each D_i is a disjunction of literals, called a *fundamental disjunction*.
- Examples:
  - $(A \wedge B) \vee (\neg A \wedge C \wedge \neg D)$ is a DNF.
  - $(A \vee B) \wedge (\neg A \vee C) \wedge (\neg C \vee \neg D)$ is a CNF.
  - A, B, $A \vee \neg B$, and $A \wedge \neg B$ are each both DNF and CNF. Why?

Any wff has a DNF and a CNF

For any propositional variable A we have True $\equiv A \vee \neg A$ and False $\equiv A \wedge \neg A$. Both forms are DNF and CNF. For other wffs use basic equivalences to:

1. remove conditionals
2. move negations to the right, and
3. transform into required form, simplifying when desired.

Example

Transform $(A \to B \vee C) \to (A \wedge D)$ into first DNF, then CNF.

$$
\begin{array}{ll}
\equiv \neg(A \to B \vee C) \vee (A \wedge D) & (X \to Y \equiv \neg X \vee Y) \\
\equiv (A \wedge \neg(B \vee C)) \vee (A \wedge D) & (\neg(X \to Y) \equiv X \wedge \neg Y) \\
\equiv (A \wedge \neg B \wedge \neg C) \vee (A \wedge D) & (\neg(X \vee Y) \equiv \neg X \wedge \neg Y) \text{ (DNF)} \\
\equiv ((A \wedge \neg B \wedge \neg C) \vee A) \wedge & \\
\qquad ((A \wedge \neg B \wedge \neg C) \vee D) & \text{(distribute } \vee \text{ over } \wedge) \\
\equiv A \wedge ((A \wedge \neg B \wedge \neg C) \vee D) & \text{(absorption)} \\
\equiv A \wedge (A \vee D) \wedge (\neg B \vee D) \wedge & \\
\qquad (\neg C \vee D) & \text{(distribute } \vee \text{ over } \wedge) \text{ (CNF)} \\
\equiv A \wedge (\neg B \vee D) \wedge (\neg C \vee D) & \text{(absorption) (CNF)}
\end{array}
$$

CNF and DNF practice problem

Transform $(A \wedge B) \vee \neg(C \rightarrow D)$ into DNF and into CNF.

Every Truth Function is a WFF

- A *truth function* is a function whose arguments and results take values in {true, false}. So a truth function can be represented by a truth table. The task is to find a wff with the same truth table. We can construct both a DNF and a CNF.
- Technique: To construct a DNF, take each line of the table with a true value and construct a fundamental conjunction that is true only on that line. To construct a CNF, take each line with a false value and construct a fundamental disjunction that is false only on that line.

Truth function example

- Example: Let f be defined by $f(A, B) =$ if $A = B$ then True else False.
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A	B	$f(A,B)$	(DNF Parts)	(CNF Parts)
T	T	T	$A \wedge B$	
T	F	F		$\neg A \vee B$
F	T	F		$A \vee \neg B$
F	F	T	$\neg A \wedge \neg B$	

- So $f(A, B)$ can be written as follows:
  - $f(A, B) = (A \wedge B) \vee (\neg A \wedge \neg B)$ (DNF)
  - $f(A, B) = (\neg A \vee B) \wedge (A \vee \neg B)$ (CNF)

Complete Sets of Connectives

A set S of connectives is *complete* if every wff is equivalent to a wff constructed from S. So $\{\neg, \wedge, \vee, \rightarrow\}$ is complete by definition. Each of the following sets is a complete set of connectives.

- $\{\neg, \wedge, \vee\}$
- $\{\neg, \wedge\}$
- $\{\neg, \vee\}$
- $\{\neg, \rightarrow\}$
- $\{\text{False}, \rightarrow\}$
- $\{\text{NAND}\}$
- $\{\text{NOR}\}$