# Section 6.3

# Formal Reasoning

- A *formal proof* (or derivation) is a sequence of wffs, where each wff is either a premise or the result of applying a proof rule to certain previous wffs in the sequence.

# Basic Proof Rules

$\frac{A, B}{A \wedge B}$ Conjunction (Conj)

$\frac{A \wedge B}{A}$ $\frac{A \wedge B}{B}$ Simplification (Simp)

$\frac{A}{A \vee B}$ $\frac{B}{A \vee B}$ Addition (Add)

$\frac{A \vee B, \neg A}{B}$ $\frac{A \vee B, \neg B}{A}$ Disjunctive Syllogism (DS)

# More Basic Proof Rules

$\frac{A \rightarrow B, A}{B}$ Modus Ponens (MP)

$\frac{\text{From } A, \text{derive } B}{A \rightarrow B}$ Conditional Proof (CP)

$\frac{\neg\neg A}{A}$ $\frac{A}{\neg\neg A}$ Double Negation (DN)

$\frac{A, \neg A}{\text{False}}$ Contradiction (Contr)

$\frac{\text{From } \neg A, \text{derive False}}{A}$ Indirect Proof (IP)

# First example proof

- Put each wff on a numbered line along with a reason. Use the letter P for a premise and follow the proof with QED.
- Example: Prove that the argument with the premises $A \vee C \rightarrow D$, $\neg B$, $A \vee B$ and conclusion $D$ is valid.

$$1.\ A \vee C \rightarrow D \quad \text{P}$$

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| | | |
|---|---|---|
| 1. | $A \vee C \to D$ | P |
| 2. | $\neg B$ | P |

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- Example: Prove that the argument with the premises $A \vee C \to D$, $\neg B$, $A \vee B$ and conclusion $D$ is valid.

| | | |
|---|---|---|
| 1. | $A \vee C \to D$ | P |
| 2. | $\neg B$ | P |
| 3. | $A \vee B$ | P |

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- Example: Prove that the argument with the premises $A \vee C \to D$, $\neg B$, $A \vee B$ and conclusion $D$ is valid.

| | | |
|---|---|---|
| 1. | $A \vee C \to D$ | P |
| 2. | $\neg B$ | P |
| 3. | $A \vee B$ | P |
| 4. | $A$ | 2,3,DS |

# First example proof

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- Example: Prove that the argument with the premises $A \vee C \to D$, $\neg B$, $A \vee B$ and conclusion $D$ is valid.

| | | |
|---|---|---|
| 1. | $A \vee C \to D$ | P |
| 2. | $\neg B$ | P |
| 3. | $A \vee B$ | P |
| 4. | $A$ | 2,3,DS |
| 5. | $A \vee C$ | 4, Add |

# First example proof

- Put each wff on a numbered line along with a reason. Use the letter P for a premise and follow the proof with QED.
- Example: Prove that the argument with the premises $A \vee C \to D$, $\neg B$, $A \vee B$ and conclusion $D$ is valid.

| | | |
|---|---|---|
| 1. | $A \vee C \to D$ | P |
| 2. | $\neg B$ | P |
| 3. | $A \vee B$ | P |
| 4. | $A$ | 2,3,DS |
| 5. | $A \vee C$ | 4, Add |
| 6. | $D$ | 1,5,MP |

# First example proof

- Put each wff on a numbered line along with a reason. Use the letter P for a premise and follow the proof with QED.
- Example: Prove that the argument with the premises $A \vee C \to D$, $\neg B$, $A \vee B$ and conclusion $D$ is valid.

| | |
|---|---|
| 1. $A \vee C \to D$ | P |
| 2. $\neg B$ | P |
| 3. $A \vee B$ | P |
| 4. $A$ | 2,3,DS |
| 5. $A \vee C$ | 4, Add |
| 6. $D$ | 1,5,MP |
| QED | |

# Using CP

- If a proof consists of a derivation from a premise $A$ to a conclusion $B$ that does not contain any uses of CP or IP, then we can apply CP to obtain a tautology $A \rightarrow B$. The reason we obtain a tautology is that the proof rules used in the derivation are valid arguments. So the truth of A implies the truth of B, which tells us that $A \rightarrow B$ is a tautology.
- When using CP in this way, instead of writing $A \rightarrow B$, we'll write QED along with the line numbers of the derivation followed by CP.

# CP proof

Example: Prove that $(A \vee C \to D) \wedge \neg B \wedge (A \vee B) \to D$ is a tautology.

1. $A \vee C \to D$ P

# CP proof

Example: Prove that $(A \vee C \to D) \wedge \neg B \wedge (A \vee B) \to D$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee C \to D$ | P |
| 2. | $\neg B$ | P |

# CP proof

Example: Prove that $(A \vee C \to D) \wedge \neg B \wedge (A \vee B) \to D$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee C \to D$ | P |
| 2. | $\neg B$ | P |
| 3. | $A \vee B$ | P |

# CP proof

Example: Prove that $(A \vee C \to D) \wedge \neg B \wedge (A \vee B) \to D$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee C \to D$ | P |
| 2. | $\neg B$ | P |
| 3. | $A \vee B$ | P |
| 4. | $A$ | 2,3,DS |

# CP proof

Example: Prove that $(A \vee C \to D) \wedge \neg B \wedge (A \vee B) \to D$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee C \to D$ | P |
| 2. | $\neg B$ | P |
| 3. | $A \vee B$ | P |
| 4. | $A$ | 2,3,DS |
| 5. | $A \vee C$ | 4, Add |

# CP proof

Example: Prove that $(A \vee C \to D) \wedge \neg B \wedge (A \vee B) \to D$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee C \to D$ | P |
| 2. | $\neg B$ | P |
| 3. | $A \vee B$ | P |
| 4. | $A$ | 2,3,DS |
| 5. | $A \vee C$ | 4, Add |
| 6. | $D$ | 1,5,MP |

# CP proof

Example: Prove that $(A \vee C \rightarrow D) \wedge \neg B \wedge (A \vee B) \rightarrow D$ is a tautology.

| | |
|---|---|
| 1. $A \vee C \rightarrow D$ | P |
| 2. $\neg B$ | P |
| 3. $A \vee B$ | P |
| 4. $A$ | 2,3,DS |
| 5. $A \vee C$ | 4, Add |
| 6. $D$ | 1,5,MP |
| QED | 1-6,CP. |

# Another CP proof

Example: Prove $(A \vee B \to C \wedge D) \wedge A \wedge (C \to E) \to D \wedge E$ is a tautology.

1. $A \vee B \to C \wedge D$    P

# Another CP proof

Example: Prove $(A \vee B \to C \wedge D) \wedge A \wedge (C \to E) \to D \wedge E$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C \wedge D$ | P |
| 2. | $A$ | P |

# Another CP proof

Example: Prove $(A \vee B \to C \wedge D) \wedge A \wedge (C \to E) \to D \wedge E$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C \wedge D$ | P |
| 2. | $A$ | P |
| 3. | $C \to E$ | P |

# Another CP proof

Example: Prove $(A \vee B \to C \wedge D) \wedge A \wedge (C \to E) \to D \wedge E$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C \wedge D$ | P |
| 2. | $A$ | P |
| 3. | $C \to E$ | P |
| 4. | $A \vee B$ | 2,Add |

# Another CP proof

Example: Prove $(A \vee B \to C \wedge D) \wedge A \wedge (C \to E) \to D \wedge E$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C \wedge D$ | P |
| 2. | $A$ | P |
| 3. | $C \to E$ | P |
| 4. | $A \vee B$ | 2,Add |
| 5. | $C \wedge D$ | 1,4, MP |

# Another CP proof

Example: Prove $(A \vee B \to C \wedge D) \wedge A \wedge (C \to E) \to D \wedge E$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C \wedge D$ | P |
| 2. | $A$ | P |
| 3. | $C \to E$ | P |
| 4. | $A \vee B$ | 2,Add |
| 5. | $C \wedge D$ | 1,4, MP |
| 6. | $C$ | 5, Simp |

# Another CP proof

Example: Prove $(A \vee B \to C \wedge D) \wedge A \wedge (C \to E) \to D \wedge E$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C \wedge D$ | P |
| 2. | $A$ | P |
| 3. | $C \to E$ | P |
| 4. | $A \vee B$ | 2,Add |
| 5. | $C \wedge D$ | 1,4, MP |
| 6. | $C$ | 5, Simp |
| 7. | $E$ | 3,6, MP |

# Another CP proof

Example: Prove $(A \vee B \to C \wedge D) \wedge A \wedge (C \to E) \to D \wedge E$ is a tautology.

| | |
|---|---|
| 1. $A \vee B \to C \wedge D$ | P |
| 2. $A$ | P |
| 3. $C \to E$ | P |
| 4. $A \vee B$ | 2,Add |
| 5. $C \wedge D$ | 1,4, MP |
| 6. $C$ | 5, Simp |
| 7. $E$ | 3,6, MP |
| 8. $D$ | 5, Simp |

# Another CP proof

Example: Prove $(A \vee B \rightarrow C \wedge D) \wedge A \wedge (C \rightarrow E) \rightarrow D \wedge E$ is a tautology.

| | |
|---|---|
| 1. $A \vee B \rightarrow C \wedge D$ | P |
| 2. $A$ | P |
| 3. $C \rightarrow E$ | P |
| 4. $A \vee B$ | 2,Add |
| 5. $C \wedge D$ | 1,4, MP |
| 6. $C$ | 5, Simp |
| 7. $E$ | 3,6, MP |
| 8. $D$ | 5, Simp |
| 9. $D \wedge E$ | 7,8, Conj |

# Another CP proof

Example: Prove $(A \vee B \rightarrow C \wedge D) \wedge A \wedge (C \rightarrow E) \rightarrow D \wedge E$ is a tautology.

| | |
|---|---|
| 1. $A \vee B \rightarrow C \wedge D$ | P |
| 2. $A$ | P |
| 3. $C \rightarrow E$ | P |
| 4. $A \vee B$ | 2,Add |
| 5. $C \wedge D$ | 1,4, MP |
| 6. $C$ | 5, Simp |
| 7. $E$ | 3,6, MP |
| 8. $D$ | 5, Simp |
| 9. $D \wedge E$ | 7,8, Conj |
| QED | 1-9, CP. |

# Subproofs

- A *subproof* is a proof that is part of another proof. It always starts with a new premise and always ends by applying CP or IP to the derivation from that premise. When this happens, the premise is *discharged* and the wffs from the derivation become inactive.
- Indent the statements of the subproof and write down the result of CP or IP without indentation.
- Example: We'll prove that $(A \vee B) \to (\neg B \to A)$ is a tautology.

| | |
|---|---|
| 1. $A \vee B$ | P |
| 2. $\quad \neg B$ | P [for $\neg B \to A$] |
| 3. $\quad A$ | 1,2,DS |
| 4. $\neg B \to A$ | 2,3, CP |
| QED | 1,4,CP. |

# Another subproof example

Prove: $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$ is a tautology.

1. $A \vee B \to C$ P

# Another subproof example

Prove: $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C$ | P |
| 2. | $\quad A$ | P [for $A \to C$] |

# Another subproof example

Prove: $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C$ | P |
| 2. | $\quad A$ | P [for $A \to C$] |
| 3. | $\quad A \vee B$ | 2,Add |

# Another subproof example

Prove: $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C$ | P |
| 2. | $\quad A$ | P [for $A \to C$] |
| 3. | $\quad A \vee B$ | 2,Add |
| 4. | $\quad C$ | 1,3,MP |

# Another subproof example

Prove: $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C$ | P |
| 2. | $\quad A$ | P [for $A \to C$] |
| 3. | $\quad A \vee B$ | 2,Add |
| 4. | $\quad C$ | 1,3,MP |
| 5. | $A \to C$ | 2-4, CP |

# Another subproof example

Prove: $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C$ | P |
| 2. | $\quad A$ | P [for $A \to C$] |
| 3. | $\quad A \vee B$ | 2,Add |
| 4. | $\quad C$ | 1,3,MP |
| 5. | $A \to C$ | 2-4, CP |
| 6. | $\quad B$ | P [for $B \to C$] |

# Another subproof example

Prove: $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C$ | P |
| 2. | $\quad A$ | P [for $A \to C$] |
| 3. | $\quad A \vee B$ | 2,Add |
| 4. | $\quad C$ | 1,3,MP |
| 5. | $A \to C$ | 2-4, CP |
| 6. | $\quad B$ | P [for $B \to C$] |
| 7. | $\quad A \vee B$ | 6, Add |

# Another subproof example

Prove: $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C$ | P |
| 2. | $\quad A$ | P [for $A \to C$] |
| 3. | $\quad A \vee B$ | 2,Add |
| 4. | $\quad C$ | 1,3,MP |
| 5. | $A \to C$ | 2-4, CP |
| 6. | $\quad B$ | P [for $B \to C$] |
| 7. | $\quad A \vee B$ | 6, Add |
| 8. | $\quad C$ | 1,7,MP |

# Another subproof example

Prove: $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C$ | P |
| 2. | $\quad A$ | P [for $A \to C$] |
| 3. | $\quad A \vee B$ | 2,Add |
| 4. | $\quad C$ | 1,3,MP |
| 5. | $A \to C$ | 2-4, CP |
| 6. | $\quad B$ | P [for $B \to C$] |
| 7. | $\quad A \vee B$ | 6, Add |
| 8. | $\quad C$ | 1,7,MP |
| 9. | $B \to C$ | 6-8, CP |

# Another subproof example

Prove: $(A \vee B \to C) \to (A \to C) \wedge (B \to C)$ is a tautology.

| | | |
|---|---|---|
| 1. | $A \vee B \to C$ | P |
| 2. | $\quad A$ | P [for $A \to C$] |
| 3. | $\quad A \vee B$ | 2,Add |
| 4. | $\quad C$ | 1,3,MP |
| 5. | $A \to C$ | 2-4, CP |
| 6. | $\quad B$ | P [for $B \to C$] |
| 7. | $\quad A \vee B$ | 6, Add |
| 8. | $\quad C$ | 1,7,MP |
| 9. | $B \to C$ | 6-8, CP |
| 10. | $(A \to C) \wedge (B \to C)$ | 5,9,Conj |

# Another subproof example

Prove: $(A \lor B \to C) \to (A \to C) \land (B \to C)$ is a tautology.

| | |
|---|---|
| 1. $A \lor B \to C$ | P |
| 2. $\quad A$ | P [for $A \to C$] |
| 3. $\quad A \lor B$ | 2,Add |
| 4. $\quad C$ | 1,3,MP |
| 5. $A \to C$ | 2-4, CP |
| 6. $\quad B$ | P [for $B \to C$] |
| 7. $\quad A \lor B$ | 6, Add |
| 8. $\quad C$ | 1,7,MP |
| 9. $B \to C$ | 6-8, CP |
| 10. $(A \to C) \land (B \to C)$ | 5,9,Conj |
| QED | 1,5,9-10,CP. |

# An Indirect Proof

- If a proof consists of a derivation from a premise $\neg A$ to the conclusion False, then we could apply CP to obtain $\neg A \rightarrow$ False. But we also know that $A \equiv \neg A \rightarrow$ False. So the result of the derivation is A. This is the IP rule.
- Example: We'll prove the tautology $\neg(A \wedge \neg A)$.

$$1.\ \neg\neg(A \wedge \neg A) \quad \text{P [for } \neg(A \wedge \neg A)]$$

# An Indirect Proof

- If a proof consists of a derivation from a premise $\neg A$ to the conclusion False, then we could apply CP to obtain $\neg A \to$ False. But we also know that $A \equiv \neg A \to$ False. So the result of the derivation is A. This is the IP rule.
- Example: We'll prove the tautology $\neg(A \wedge \neg A)$.

| | | |
|---|---|---|
| 1. | $\neg\neg(A \wedge \neg A)$ | P [for $\neg(A \wedge \neg A)$] |
| 2. | $A \wedge \neg A$ | 1,DN |

# An Indirect Proof

- If a proof consists of a derivation from a premise $\neg A$ to the conclusion False, then we could apply CP to obtain $\neg A \rightarrow$ False. But we also know that $A \equiv \neg A \rightarrow$ False. So the result of the derivation is A. This is the IP rule.
- Example: We'll prove the tautology $\neg(A \wedge \neg A)$.

| | | |
|---|---|---|
| 1. | $\neg\neg(A \wedge \neg A)$ | P [for $\neg(A \wedge \neg A)$] |
| 2. | $A \wedge \neg A$ | 1,DN |
| 3. | $A$ | 2,Simp |

# An Indirect Proof

- If a proof consists of a derivation from a premise $\neg A$ to the conclusion False, then we could apply CP to obtain $\neg A \rightarrow$ False. But we also know that $A \equiv \neg A \rightarrow$ False. So the result of the derivation is A. This is the IP rule.
- Example: We'll prove the tautology $\neg(A \wedge \neg A)$.

| | | |
|---|---|---|
| 1. | $\neg\neg(A \wedge \neg A)$ | P [for $\neg(A \wedge \neg A)$] |
| 2. | $A \wedge \neg A$ | 1,DN |
| 3. | $A$ | 2,Simp |
| 4. | $\neg A$ | 2,Simp |

# An Indirect Proof

- If a proof consists of a derivation from a premise $\neg A$ to the conclusion False, then we could apply CP to obtain $\neg A \rightarrow$ False. But we also know that $A \equiv \neg A \rightarrow$ False. So the result of the derivation is A. This is the IP rule.
- Example: We'll prove the tautology $\neg(A \wedge \neg A)$.

| | | |
|---|---|---|
| 1. | $\neg\neg(A \wedge \neg A)$ | P [for $\neg(A \wedge \neg A)$] |
| 2. | $A \wedge \neg A$ | 1,DN |
| 3. | $A$ | 2,Simp |
| 4. | $\neg A$ | 2,Simp |
| 5. | False | 3,4,Contr |

# An Indirect Proof

- If a proof consists of a derivation from a premise $\neg A$ to the conclusion False, then we could apply CP to obtain $\neg A \rightarrow$ False. But we also know that $A \equiv \neg A \rightarrow$ False. So the result of the derivation is A. This is the IP rule.
- Example: We'll prove the tautology $\neg(A \wedge \neg A)$.

| | |
|---|---|
| 1. $\neg\neg(A \wedge \neg A)$ | P [for $\neg(A \wedge \neg A)$] |
| 2. $A \wedge \neg A$ | 1,DN |
| 3. $A$ | 2,Simp |
| 4. $\neg A$ | 2,Simp |
| 5. False | 3,4,Contr |
| QED | 1-5,IP. |

# IP proofs

- IP is most often used in a subproof setting when proving a conditional of the form $V \to W$. Start with V as a premise for a CP proof. Then start an IP subproof with premise $\neg W$. When a contradiction is reached, we obtain $W$ by IP. Then CP gives the result $V \to W$.
- As with CP subproofs, the result of IP is written with no indentation.

# IP Example

Prove the tautology: $(A \to B) \wedge (A \vee B) \to B$.

1. $A \to B$ P

# IP Example

Prove the tautology: $(A \to B) \wedge (A \vee B) \to B$.

| | | |
|---|---|---|
| 1. | $A \to B$ | P |
| 2. | $A \vee B$ | P |

# IP Example

Prove the tautology: $(A \to B) \wedge (A \vee B) \to B$.

| | | |
|---|---|---|
| 1. | $A \to B$ | P |
| 2. | $A \vee B$ | P |
| 3. | $\neg B$ | P [for $B$] |

# IP Example

Prove the tautology: $(A \to B) \wedge (A \vee B) \to B$.

| | | |
|---|---|---|
| 1. | $A \to B$ | P |
| 2. | $A \vee B$ | P |
| 3. | $\neg B$ | P [for $B$] |
| 4. | $A$ | 2,3,DS |

# IP Example

Prove the tautology: $(A \to B) \wedge (A \vee B) \to B$.

| | | |
|---|---|---|
| 1. | $A \to B$ | P |
| 2. | $A \vee B$ | P |
| 3. | $\neg B$ | P [for $B$] |
| 4. | $A$ | 2,3,DS |
| 5. | $B$ | 1,4,MP |

# IP Example

Prove the tautology: $(A \to B) \wedge (A \vee B) \to B$.

| | | |
|---|---|---|
| 1. | $A \to B$ | P |
| 2. | $A \vee B$ | P |
| 3. | $\neg B$ | P [for $B$] |
| 4. | $A$ | 2,3,DS |
| 5. | $B$ | 1,4,MP |
| 6. | False | 3,5,Contr |

# IP Example

Prove the tautology: $(A \to B) \land (A \lor B) \to B$.

| | | |
|---|---|---|
| 1. | $A \to B$ | P |
| 2. | $A \lor B$ | P |
| 3. | $\quad \neg B$ | P [for $B$] |
| 4. | $\quad A$ | 2,3,DS |
| 5. | $\quad B$ | 1,4,MP |
| 6. | $\quad$ False | 3,5,Contr |
| 7. | $B$ | 3-6, IP |

# IP Example

Prove the tautology: $(A \to B) \land (A \lor B) \to B$.

| | | |
|---|---|---|
| 1. | $A \to B$ | P |
| 2. | $A \lor B$ | P |
| 3. | $\quad \neg B$ | P [for $B$] |
| 4. | $\quad A$ | 2,3,DS |
| 5. | $\quad B$ | 1,4,MP |
| 6. | $\quad$ False | 3,5,Contr |
| 7. | $B$ | 3-6, IP |
| QED | | 1-2,7,CP. |

# Another IP Example

We'll prove the converse of $(A \vee B) \to (\neg B \to A)$. Proof of $(\neg B \to A) \to (A \vee B)$.

1. $\neg B \to A$ P

# Another IP Example

We'll prove the converse of $(A \vee B) \to (\neg B \to A)$. Proof of $(\neg B \to A) \to (A \vee B)$.

| | | |
|---|---|---|
| 1. | $\neg B \to A$ | P |
| 2. | $\quad \neg(A \vee B)$ | P [for $A \vee B$] |

# Another IP Example

We'll prove the converse of $(A \vee B) \rightarrow (\neg B \rightarrow A)$. Proof of $(\neg B \rightarrow A) \rightarrow (A \vee B)$.

| | | |
|---|---|---|
| 1. | $\neg B \rightarrow A$ | P |
| 2. | $\quad \neg(A \vee B)$ | P [for $A \vee B$] |
| 3. | $\qquad \neg B$ | P [for B] |

# Another IP Example

We'll prove the converse of $(A \vee B) \rightarrow (\neg B \rightarrow A)$. Proof of $(\neg B \rightarrow A) \rightarrow (A \vee B)$.

| | | |
|---|---|---|
| 1. | $\neg B \rightarrow A$ | P |
| 2. | $\quad \neg(A \vee B)$ | P [for $A \vee B$] |
| 3. | $\qquad \neg B$ | P [for B] |
| 4. | $\qquad A$ | 1,3,MP |

# Another IP Example

We'll prove the converse of $(A \vee B) \to (\neg B \to A)$. Proof of $(\neg B \to A) \to (A \vee B)$.

| | | |
|---|---|---|
| 1. | $\neg B \to A$ | P |
| 2. | $\quad \neg(A \vee B)$ | P [for $A \vee B$] |
| 3. | $\qquad \neg B$ | P [for B] |
| 4. | $\qquad A$ | 1,3,MP |
| 5. | $\qquad A \vee B$ | 4,Add |

# Another IP Example

We'll prove the converse of $(A \vee B) \rightarrow (\neg B \rightarrow A)$. Proof of $(\neg B \rightarrow A) \rightarrow (A \vee B)$.

| | | |
|---|---|---|
| 1. | $\neg B \rightarrow A$ | P |
| 2. | $\neg(A \vee B)$ | P [for $A \vee B$] |
| 3. | $\neg B$ | P [for B] |
| 4. | $A$ | 1,3,MP |
| 5. | $A \vee B$ | 4,Add |
| 6. | False | 2,5,Contr |

# Another IP Example

We'll prove the converse of $(A \vee B) \rightarrow (\neg B \rightarrow A)$. Proof of $(\neg B \rightarrow A) \rightarrow (A \vee B)$.

| | | |
|---|---|---|
| 1. | $\neg B \rightarrow A$ | P |
| 2. | $\quad \neg(A \vee B)$ | P [for $A \vee B$] |
| 3. | $\qquad \neg B$ | P [for B] |
| 4. | $\qquad A$ | 1,3,MP |
| 5. | $\qquad A \vee B$ | 4,Add |
| 6. | $\qquad$ False | 2,5,Contr |
| 7. | $\quad B$ | 3-6,IP |

# Another IP Example

We'll prove the converse of $(A \vee B) \to (\neg B \to A)$. Proof of $(\neg B \to A) \to (A \vee B)$.

| | | |
|---|---|---|
| 1. | $\neg B \to A$ | P |
| 2. | $\quad \neg(A \vee B)$ | P [for $A \vee B$] |
| 3. | $\qquad \neg B$ | P [for B] |
| 4. | $\qquad A$ | 1,3,MP |
| 5. | $\qquad A \vee B$ | 4,Add |
| 6. | $\qquad$ False | 2,5,Contr |
| 7. | $\quad B$ | 3-6,IP |
| 8. | $\quad A \vee B$ | 7,Add |

# Another IP Example

We'll prove the converse of $(A \vee B) \to (\neg B \to A)$. Proof of $(\neg B \to A) \to (A \vee B)$.

| | | |
|---|---|---|
| 1. | $\neg B \to A$ | P |
| 2. | $\quad \neg(A \vee B)$ | P [for $A \vee B$] |
| 3. | $\qquad \neg B$ | P [for B] |
| 4. | $\qquad A$ | 1,3,MP |
| 5. | $\qquad A \vee B$ | 4,Add |
| 6. | $\qquad$ False | 2,5,Contr |
| 7. | $\quad B$ | 3-6,IP |
| 8. | $\quad A \vee B$ | 7,Add |
| 9. | $\quad$ False | 2,8,Contr |

# Another IP Example

We'll prove the converse of $(A \vee B) \to (\neg B \to A)$. Proof of $(\neg B \to A) \to (A \vee B)$.

| | | |
|---|---|---|
| 1. | $\neg B \to A$ | P |
| 2. | $\quad \neg(A \vee B)$ | P [for $A \vee B$] |
| 3. | $\qquad \neg B$ | P [for B] |
| 4. | $\qquad A$ | 1,3,MP |
| 5. | $\qquad A \vee B$ | 4,Add |
| 6. | $\qquad$ False | 2,5,Contr |
| 7. | $\quad B$ | 3-6,IP |
| 8. | $\quad A \vee B$ | 7,Add |
| 9. | $\quad$ False | 2,8,Contr |
| 10. | $A \vee B$ | 2,7-9,IP |

# Another IP Example

We'll prove the converse of $(A \vee B) \to (\neg B \to A)$. Proof of $(\neg B \to A) \to (A \vee B)$.

| | | |
|---|---|---|
| 1. | $\neg B \to A$ | P |
| 2. | $\quad \neg(A \vee B)$ | P [for $A \vee B$] |
| 3. | $\qquad \neg B$ | P [for B] |
| 4. | $\qquad A$ | 1,3,MP |
| 5. | $\qquad A \vee B$ | 4,Add |
| 6. | $\qquad$ False | 2,5,Contr |
| 7. | $\quad B$ | 3-6,IP |
| 8. | $\quad A \vee B$ | 7,Add |
| 9. | $\quad$ False | 2,8,Contr |
| 10. | $A \vee B$ | 2,7-9,IP |
| QED | | 1,10,CP. |

# Derived Proof Rules

$\frac{A \rightarrow B, \neg B}{\neg A}$ Modus Tollens (MT)

$\frac{A \vee B, A \rightarrow C, B \rightarrow C}{C}$ Proof by Cases (Cases)

$\frac{A \rightarrow B, B \rightarrow C}{A \rightarrow C}$ Hypothetical Syllogism (HS)

$\frac{A \vee B, A \rightarrow C, B \rightarrow D}{C \vee D}$ Constructive Dilemma (CD)

# Proofs using derived rules

We'll give two proof of the tautology $(A \to C) \wedge (B \to C) \to (A \vee B \to C)$.

1. $A \to C$ P

# Proofs using derived rules

We'll give two proof of the tautology
$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$.

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |

# Proofs using derived rules

We'll give two proof of the tautology
$(A \to C) \land (B \to C) \to (A \lor B \to C)$.

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $A \lor B$ | P [for $A \lor B \to C$] |

# Proofs using derived rules

We'll give two proof of the tautology
$(A \to C) \land (B \to C) \to (A \lor B \to C)$.

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $\quad A \lor B$ | P [for $A \lor B \to C$] |
| 4. | $\quad C$ | 1,2,3,Cases |

# Proofs using derived rules

We'll give two proof of the tautology
$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$.

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $\quad A \vee B$ | P [for $A \vee B \to C$] |
| 4. | $\quad C$ | 1,2,3,Cases |
| 5. | $A \vee B \to C$ | 3-4, CP |

# Proofs using derived rules

We'll give two proof of the tautology
$(A \to C) \wedge (B \to C) \to (A \vee B \to C)$.

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $\quad A \vee B$ | P [for $A \vee B \to C$] |
| 4. | $\quad C$ | 1,2,3,Cases |
| 5. | $A \vee B \to C$ | 3-4, CP |
| QED | | 1-2,5,CP |

# Second proof of the same tautology

1. $A \to C$ P

# Second proof of the same tautology

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |

# Second proof of the same tautology

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $\quad A \vee B$ | P [for $A \vee B \to C$] |

# Second proof of the same tautology

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $A \vee B$ | P [for $A \vee B \to C$] |
| 4. | $\neg C$ | P [for C] |

# Second proof of the same tautology

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $\quad A \lor B$ | P [for $A \lor B \to C$] |
| 4. | $\quad\quad \neg C$ | P [for C] |
| 5. | $\quad\quad \neg A$ | 1,4,MT |

## Second proof of the same tautology

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $A \vee B$ | P [for $A \vee B \to C$] |
| 4. | $\neg C$ | P [for C] |
| 5. | $\neg A$ | 1,4,MT |
| 6. | $B$ | 3,5,DS |

## Second proof of the same tautology

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $A \lor B$ | P [for $A \lor B \to C$] |
| 4. | $\neg C$ | P [for C] |
| 5. | $\neg A$ | 1,4,MT |
| 6. | $B$ | 3,5,DS |
| 7. | $\neg B$ | 2,4,MT |

# Second proof of the same tautology

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $A \lor B$ | P [for $A \lor B \to C$] |
| 4. | $\neg C$ | P [for C] |
| 5. | $\neg A$ | 1,4,MT |
| 6. | $B$ | 3,5,DS |
| 7. | $\neg B$ | 2,4,MT |
| 8. | False | 6,7,Contr |

# Second proof of the same tautology

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $A \lor B$ | P [for $A \lor B \to C$] |
| 4. | $\neg C$ | P [for C] |
| 5. | $\neg A$ | 1,4,MT |
| 6. | $B$ | 3,5,DS |
| 7. | $\neg B$ | 2,4,MT |
| 8. | False | 6,7,Contr |
| 9. | $C$ | 4-8,IP |

# Second proof of the same tautology

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $A \vee B$ | P [for $A \vee B \to C$] |
| 4. | $\neg C$ | P [for C] |
| 5. | $\neg A$ | 1,4,MT |
| 6. | $B$ | 3,5,DS |
| 7. | $\neg B$ | 2,4,MT |
| 8. | False | 6,7,Contr |
| 9. | $C$ | 4-8,IP |
| 10. | $A \vee B \to C$ | 3,9,CP |

# Second proof of the same tautology

| | | |
|---|---|---|
| 1. | $A \to C$ | P |
| 2. | $B \to C$ | P |
| 3. | $\quad A \vee B$ | P [for $A \vee B \to C$] |
| 4. | $\qquad \neg C$ | P [for C] |
| 5. | $\qquad \neg A$ | 1,4,MT |
| 6. | $\qquad B$ | 3,5,DS |
| 7. | $\qquad \neg B$ | 2,4,MT |
| 8. | $\qquad$ False | 6,7,Contr |
| 9. | $\quad C$ | 4-8,IP |
| 10. | $A \vee B \to C$ | 3,9,CP |
| QED | | 1-2,10,CP. |