First-Order Predicate Calculus

Predicates

- A *predicate* describes a property of the subject or subjects of a sentence. If p is a predicate that describes a property of x, then we write $p(x)$.
- *Example:* if p is the "is prime" predicate and x is prime, then we can write $p(x)$ for "x is prime".
- *Example:* if q is the "is a parent of" predicate, and x is a parent of y, then we can write $q(x, y)$ for "x is the parent of y".

Quantifiers

- First-Order Logic (FOL) has *quantifiers* that let us ascribe properties to variables.
- The Existential Quantifier is best described by the phrase: "there exists an x such that $p(x)$" and is denoted by $\exists x\ p(x)$. It indicates disjunction.
  - *Example:* If $x \in \{1, 2, 3\}$ then $\exists x\ p(x) = p(1) \vee p(2) \vee p(3)$.
- The Universal Quantifier is best described by the phrase: "for every x, $p(x)$" and is denoted by $\forall x\ p(x)$. It indicates conjunction.
  - *Example:* If $x \in \{1, 2, 3\}$ then $\forall x\ p(x) = p(1) \wedge p(2) \wedge p(3)$.
- We sometimes refer to the set of possible values for x as the *universe of discourse* and our book is sometimes careless about this. Often it is obvious, but sometimes it is not.

More Quantifiers

- Quick quiz: If $x, y \in \{0, 1\}$ then $\exists x \, \forall y \; p(x, y, z) = ?$

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- An answer: $\exists x \; (p(x, 0, z) \wedge p(x, 1, z)) =$
  $(p(0, 0, z) \wedge p(0, 1, z)) \vee (p(1, 0, z) \wedge p(1, 1, z))$.

Syntax of FOL

- *terms* are nonlogical things:
  - *constants* such as: a, b, c
  - *variables* such as: x, y, z
  - function symbols applied to terms such as: $f(a), g(x, f(y)), h(c, z)$
- *atoms* are predicate symbols applied to terms, such as: $p(x), q(a, f(x)), \ldots$
- *wffs* (well-formed formulas) are either atoms or, if U and V are wffs, then the following expressions are also wffs:
  - $\neg U, U \land V, U \lor V, U \to V, \exists x\ U, \forall x\ U, and(U)$
- precedence hierarchies in the absence of parentheses:
  - $\neg, \exists x, \forall x$ (highest, group rightmost operator with smallest wff to its right)
  - $\land$
  - $\lor$
  - $\to$ (lowest, and it is left associative)

Scope

- The *scope* of $\exists x$ in $\exists x\ W$ is W. The *scope* of $\forall x$ in $\forall x\ W$ is W. An occurrence of x is *bound* if it occurs in either $\exists x$ or $\forall x$ or in their scope. Otherwise the occurrence of x is *free*.
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- Consider: $\exists x\ p(x) \rightarrow \forall y\ q(x, y)$
- last x (in $q(x, y)$) is free, all other variables bound

Interpretations

- An *interpretation* for a first-order wff consists of a nonempty set D, called the *domain*, together with an assignment of the symbols of the wff as follows:
  1. Predicate letters are assigned to relations over D.
  2. Function letters are assigned to functions over D.
  3. Constants are assigned to elements of D.
  4. Free occurrences of variables are assigned to elements of D.
- *Notation*: If x is free in W and $d \in D$, then $W(x/d)$ denotes the wff obtained from W by replacing all free occurrences of x by d. We also write $W(d) = W(x/d)$.
- *Example*: Let $W = \forall y\ (p(x, y) \to q(x))$. Then $W(x/d) = \forall y\ (p(d, y) \to q(d))$.

Truth Value of a wff

- The truth value of a wff with respect to an interpretation with domain D is obtained by recursively applying the following rules:
  1. An atom has the truth value of the proposition obtained from its interpretation.
  2. Truth values for $\neg U, U \wedge V, U \vee V, U \to V$ are obtained by applying truth tables for $\neg, \wedge, \vee, \to$ to the truth values for U and V.
  3. $\forall x\ W$ is true iff $W(x/d)$ is true for all $d \in D$.
  4. $\exists x\ W$ is true iff $W(x/d)$ is true for some $d \in D$.
- If a wff is true with respect to an interpretation I, we say the wff "is true for I". (Really)

Interpretation examples

- *Example:* Let $W = p(x)$. We can define an interpretation I by letting $D = \mathbb{N}, p(x)$ means x is odd, and $x = 4$. Then W is false for I because $W(x/4) = p(x)(x/4) = p(4) =$"4 is odd", which is false.
- If we let J be the same as I except we assign $x = 3$, then W is true for J because $W(x/3) = p(x)(x/3) = p(3) =$"3 is odd", which is true.

Models and Countermodels and Validity

- An interpretation that makes a wff true is called a model. An interpretation that makes a wff false is called a countermodel. See the previous examples.
- A wff is valid if every interpretation is a model. Otherwise the wff is invalid. A wff is unsatisfiable if every interpretation is a countermodel. Otherwise a wff is satisfiable.
- *Quiz:* Every wff has exactly two of these properties. What are the possible pairs of properties that a wff can have?

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- *Quiz:* Every wff has exactly two of these properties. What are the possible pairs of properties that a wff can have?
- *Answer:* {valid, satisfiable}, {invalid, unsatisfiable}, {invalid, satisfiable}

Proofs of Validity or Unsatisfiability

We can't check every interpretation (there are too many). So we need to reason informally with interpretations.

- *Example:* $\forall x\ (p(x) \to p(x))$ is valid because $p(x) \to p(x)$ is true for all interpretations.
- *Example:* $\forall x\ (p(x) \wedge \neg p(x))$ is unsatisfiable because $p(x) \wedge \neg p(x)$ is always false.
- *Less simple example:* Prove the following wff is valid: $\exists x\ (A(x) \wedge B(x)) \to \exists x\ A(x) \wedge \exists x B(x)$

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- *Less simple example:* Prove the following wff is valid: $\exists x\,(A(x) \wedge B(x)) \to \exists x\,A(x) \wedge \exists x B(x)$
- *Direct proof:* Let I be an interpretation with domain D for the wff and assume that the antecedent is true for I. Then $A(d) \wedge B(d)$ is true for I for some $d \in D$. So both $A(d)$ and $B(d)$ are true for I. Therefore, both $\exists x\,A(x)$ and $\exists x\,B(x)$ are true for I. So the consequent is true for I. Thus I is a model for the wff. Since I was an arbitrary interpretation for the wff, every interpretation for the wff is a model. Therefore, the wff is valid. QED.

Another informal example

- *Quiz:* Is $\exists x\, A(x) \wedge \exists x\, B(x) \rightarrow \exists x\, (A(x) \wedge B(x))$ valid?

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- *Quiz:* Is $\exists x\, A(x) \wedge \exists x\, B(x) \rightarrow \exists x\, (A(x) \wedge B(x))$ valid?
- *Answers:* No. For example, let $D = \mathbb{N}$, $A(x)$ mean x is even, and $B(x)$ mean x is odd.

Decidability (Solvability)

- A problem in the form of a yes/no question is decidable if there is an algorithm that takes as input any instance of the problem and halts with the answer.
- Otherwise, the problem is undecidable. A problem is partially decidable if there is an algorithm that takes an input any instance of the problem and halts if the answer is yes, but might not halt if the answer is no.

Two famous validity problems

- *The Validity Problem for Propositional Calculus*
- The problem of determining whether a propositional wff is a tautology is *decidable*. An algorithm can build a truth table for the wff and then check it.
- *The Validity Problem for First-Order Predicate Calculus*
- The problem of determining whether a first-order wff is valid is *undecidable*, but it is *partially decidable*. Two partial decision procedures are *natural deduction* (due to Gentzen in 1935) and *resolution* (due to Robinson in 1965). We'll study natural deduction in Section 7.3 and resolution in Chapter 9.