Equivalent Formulas (Section 7.2)

Equivalence

- Two wffs A and B are equivalent, written $A \equiv B$, if they have the same truth value for every interpretation.
- *Property:* $A \equiv B$ iff $A \rightarrow B$ and $B \rightarrow A$ are both valid.

First-Order Equivalence

- Propositional equivalence gives rise to first-order equivalence.
- In other words, if two propositional wffs are equivalent and each occurrence of a propositional variable is replaced by a first order wff, then the resulting two first order wffs, called instances, are equivalent.
- *Example:* We have $\forall x\ p(x) \to \exists x\ p(x) \equiv \neg\forall x\ p(x) \vee \exists x\ p(x)$ because $A \to B \equiv \neg A \vee B$. So the first-order equivalence is an instance of the propositional equivalence by letting $A = \forall x\ p(x)$ and $B = \exists x\ p(x)$.

Basic Equivalences (1)

- (1) $\neg\forall x\ W \equiv \exists x \neg W$ and $\neg\exists x\ W \equiv \forall x\ \neg W$.
- *Proof: (of the second)* Let I be an arbitrary interpretation with domain D. Then

$$
\begin{aligned}
\neg\exists x\ W \text{ is true for } I \quad & \text{iff } \exists x\ W \text{ is false for } I \\
& \text{iff } W(x/d) \text{ is false for } I \text{ for all } d \in D \\
& \text{iff } \neg W(x/d) \text{ is true for } I \text{ for all } d \in D \\
& \text{iff } \forall x\ \neg W \text{ is true for } I.
\end{aligned}
$$

- Since I was arbitrary, the wffs are equivalent. QED.

Basic Equivalences (2)

- (2) $\forall x \forall y\ W \equiv \forall y \forall x\ W$ and $\exists x \exists y\ W \equiv \exists y \exists x\ W$.
- *Proof:* Left as an exercise for the reader.

Basic Equivalences (3)

- (3) $\exists x\ (A(x) \rightarrow B(x)) \equiv \forall x\ A(x) \rightarrow \exists x\ B(x)$
- *Proof:* Let I be an arbitrary interpretation with domain D. Assume LHS is true for I. Then $A(c) \rightarrow B(c)$ is true for I for some $c \in D$. Consider the possible values of $A(c)$. If $A(c)$ is true for I, the $B(c)$ is true for I. So $\exists x\ B(x)$ is true for I, which implies RHS is true for I. If $A(c)$ is false for I, the $\forall x\ A(X)$ is false for I, which implies RHS is true for I. So LHS $\rightarrow$ LHS is valid. Converse proof is left as an exercise.

Basic Equivalences (4)

- (4) $\exists x\ (A(x) \vee B(x)) \equiv \exists x\ A(x) \vee \exists x\ B(x)$
- *Proof:*

$$
\begin{aligned}
\exists x\ (A(x) \vee B(x)) &\equiv \exists x\ (\neg A(x) \rightarrow B(x)) && \text{(instance of prop wff)} \\
&\equiv \forall x\ \neg A(x) \rightarrow \exists x\ B(x) && (3) \\
&\equiv \neg \exists x\ A(x) \rightarrow \exists x\ B(x) && (1) \\
&\equiv \exists x\ A(x) \vee \exists x\ B(x) && \text{(instance of prop wff)} \\
&\text{QED}
\end{aligned}
$$

Basic Equivalences (5)

- (5) $\forall x\ (A(x) \wedge B(x)) \equiv \forall x\ A(x) \wedge \forall x\ B(x)$.
- *Proof:* Left as an exercise for the reader.

Restricted Equivalences (6)

- (6) (Renaming Variables) If y does not occur in $W(x)$, then $\exists x \; W(x) \equiv \exists y \; W(y)$ and $\forall x \; W(x) \equiv \forall y \; W(y)$.

Restricted Equivalences (7)

- (7) If x does not occur free in C, then
  - (a) $\forall x\ C \equiv C$ and $\exists x\ C \equiv C$
  - (b) $\forall x\ (C \vee A(x)) \equiv C \vee \forall x\ A(x)$ and $\exists x\ (C \vee A(x)) \equiv C \vee \exists x\ A(x)$.
  - (c) $\forall x\ (C \wedge A(x)) \equiv C \wedge \forall x\ A(x)$ and $\exists x\ (C \wedge A(x)) \equiv C \wedge \exists x\ A(x)$.
  - (d) $\forall x\ (C \rightarrow A(x)) \equiv C \rightarrow \forall x\ A(x)$ and $\exists x\ (C \rightarrow A(x)) \equiv C \rightarrow \exists x\ A(x)$.
  - (e) Be careful with these:
    - $\forall x\ (A(x) \rightarrow C) \equiv \exists x\ A(x) \rightarrow C$.
    - $\exists x\ (A(x) \rightarrow C) \equiv \forall x\ A(x) \rightarrow C$.

Normal Forms

A wff in prenex normal form has all quantifiers at the left end. For example: $\exists x \forall y\ (p(x) \to q(y))$.

Prenex Normal Form Algorithm:

- Rename variables to get distinct quantifier names and distinct free variable names.
- Move quantifiers left using (1), (7b), (7c), (7d) and (7e).
- *Example:*

$$
\begin{array}{ll}
q(x) \wedge \exists x\ (r(x) \to \neg \exists y\ p(x,y)) & \\
\equiv q(x) \wedge \exists z\ (r(z) \to \neg \exists y\ p(z,y)) & \text{(rename)} \\
\equiv \exists z\ (q(x) \wedge (r(z) \to \neg \exists y\ p(z,y))) & \text{(7b)} \\
\equiv \exists z\ (q(x) \wedge (r(z) \to \forall y \neg p(z,y))) & \text{(1)} \\
\equiv \exists z\ (q(x) \wedge \forall y\ (r(z) \to \neg p(z,y))) & \text{(7d)} \\
\equiv \exists z \forall y\ (q(x) \wedge (r(z) \to \neg p(z,y))) & \text{(7b)}
\end{array}
$$

Prenex CNF/DNF

- A wff in prenex normal form is in prenex CNF (or prenex DNF) if the wff to the right of the quantifiers is in CNF (or DNF), where a literal is now either an atom or its negation.
- *Example:* $p(x)$ and $\neg p(x)$ are literals.
- *Example:* $\exists z\, \forall y\, (q(x) \wedge (\neg r(z) \vee \neg p(z,y)))$ is a prenex CNF.
- *Example:* $\exists z\, \forall y\, ((q(x) \wedge \neg r(z)) \vee (q(x) \wedge \neg p(z,y)))$ is a prenex DNF.

Prenex CNF/DNF Algorithm

1. Put wff in prenex normal form.
2. Remove $\to$.
3. Move $\neg$ to the right to form literals.
4. Distribute $\vee$ over $\wedge$ and/or $\wedge$ over $\vee$ for desired form.

Example:

$$\begin{array}{ll}
\forall x \forall y \exists z\, (q(x) \vee r(z,x) \to p(z,y)) & \text{(prenex normal form)} \\
\equiv \forall x \forall y \exists z\, (\neg(q(x) \vee r(z,x)) \vee p(z,y)) & \text{(remove } \to \text{)} \\
\equiv \forall x \forall y \exists z\, ((\neg q(x) \wedge \neg r(z,x)) \vee p(z,y)) & \text{(move } \neg \text{ right)(prenex DNF} \\
\equiv \forall x \forall y \exists z\, ((\neg q(x) \vee p(z,y)) & \\
\wedge(\neg r(z,x) \vee p(z,y))) & \text{(distribute)(prenex CNF)}
\end{array}$$

Formalizing English Sentences

Some rules that usually work for English sentences are:

- $\forall x$ quantifies a conditional.
- $\exists x$ quantifies a conjunction.
- Use $\forall x$ with conditional for "all", "every", and "only".
- Use $\exists x$ with conjunction for "some", "there is", and "not all".
- Use $\forall x$ with conditional or $\neg\exists x$ with conjunction for "no A is B".
- Use $\exists x$ with conjunction or $\neg\forall x$ with conditional for "not all A's are B".

English examples

For a person x let $c(x)$ mean x is a CS major and let $s(x)$ mean x is smart. Then formalize each of the following sentences:

1. All CS majors are smart.

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5. Only CS majors are smart.

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7. Some CS majors are smart.

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7. Some CS majors are smart.
8. *Solution:* $\exists x\, (c(x) \wedge s(x))$

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For a person x let $c(x)$ mean x is a CS major and let $s(x)$ mean x is smart. Then formalize each of the following sentences:

1. All CS majors are smart.
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3. Every CS major is smart.
4. *Solution:* $\forall x\ (c(x) \to s(x))$
5. Only CS majors are smart.
6. *Solution:* $\forall x\ (s(x) \to c(x))$
7. Some CS majors are smart.
8. *Solution:* $\exists x\ (c(x) \wedge s(x))$
9. There is a CS major that is smart.

English examples

For a person x let $c(x)$ mean x is a CS major and let $s(x)$ mean x is smart. Then formalize each of the following sentences:

1. All CS majors are smart.
2. *Solution:* $\forall x\,(c(x) \rightarrow s(x))$
3. Every CS major is smart.
4. *Solution:* $\forall x\,(c(x) \rightarrow s(x))$
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6. *Solution:* $\forall x\,(s(x) \rightarrow c(x))$
7. Some CS majors are smart.
8. *Solution:* $\exists x\,(c(x) \wedge s(x))$
9. There is a CS major that is smart.
10. *Solution:* $\exists x\,(c(x) \wedge s(x))$

English examples

For a person x let $c(x)$ mean x is a CS major and let $s(x)$ mean x is smart. Then formalize each of the following sentences:

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7. Some CS majors are smart.
8. *Solution:* $\exists x\,(c(x) \wedge s(x))$
9. There is a CS major that is smart.
10. *Solution:* $\exists x\,(c(x) \wedge s(x))$
11. No CS major is smart.

English examples

For a person x let $c(x)$ mean x is a CS major and let $s(x)$ mean x is smart. Then formalize each of the following sentences:

1. All CS majors are smart.
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3. Every CS major is smart.
4. *Solution:* $\forall x\, (c(x) \to s(x))$
5. Only CS majors are smart.
6. *Solution:* $\forall x\, (s(x) \to c(x))$
7. Some CS majors are smart.
8. *Solution:* $\exists x\, (c(x) \wedge s(x))$
9. There is a CS major that is smart.
10. *Solution:* $\exists x\, (c(x) \wedge s(x))$
11. No CS major is smart.
12. *Solution:* $\forall x\, (c(x) \to \neg s(x)) \equiv \neg \exists x\, (c(x) \wedge s(x))$

English examples

For a person x let $c(x)$ mean x is a CS major and let $s(x)$ mean x is smart. Then formalize each of the following sentences:

1. All CS majors are smart.
2. *Solution:* $\forall x\ (c(x) \rightarrow s(x))$
3. Every CS major is smart.
4. *Solution:* $\forall x\ (c(x) \rightarrow s(x))$
5. Only CS majors are smart.
6. *Solution:* $\forall x\ (s(x) \rightarrow c(x))$
7. Some CS majors are smart.
8. *Solution:* $\exists x\ (c(x) \wedge s(x))$
9. There is a CS major that is smart.
10. *Solution:* $\exists x\ (c(x) \wedge s(x))$
11. No CS major is smart.
12. *Solution:* $\forall x\ (c(x) \rightarrow \neg s(x)) \equiv \neg \exists x\ (c(x) \wedge s(x))$
13. Not all CS majors are smart.
14. *Solution:* $\exists x\ c(x) \wedge \neg s(x)) \equiv \neg \forall x\ (c(x) \rightarrow s(x))$